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Mechanicam vero duplicem Veteres constituerunt: Rationalem *quae per Demonstrationes accurate procedit, & Practicam. Ad practicam spectant Artes omnes Manuales, a quibus utique* Mechanica *nomen mutuata est. Cum autem Artifices parum accurate operari soleant, fit ut* Mechanica *omnis a* Geometria *ita distinguatur, ut quicquid accuratum sit ad* Geometriam *referatur, quicquid minus accuratum ad* Mechanicam. *Attamen errores non sunt Artis sed Artificum. Qui minus accurate operatur, imperfectior est Mechanicus, & si quis accuratissime operari posset, hic foret Mechanicus omnium perfectissimus.* NEWTON

La généralité que j'embrasse, au lieu d'éblouïr nos lumieres, nous découvrira plutôt les véritables loix de la Nature dans tout leur éclat, & on y trouvera des raisons encore plus fortes, d'en admirer la beauté & la simplicité. EULER

. . . ut proinde his paucis consideratis tota haec materia redacta sit ad puram Geometriam, quod in physicis & mechanicis unice desideratum. LEIBNIZ

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An Approximation Theorem for Functionals, with Applications in Continuum Mechanics

BERNARD D. COLEMAN & WALTER NOLL

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1. Introduction

We start with a brief discussion of the physical motivation behind the mathematical considerations to be presented in Part I of this paper.

Often in theoretical physics one is concerned with a functional relationship,

$$\boldsymbol{\sigma} = \mathop{\mathfrak{F}}_{s=0}^{\infty} (\boldsymbol{g}(s)), \tag{1.1}$$

which states that the present value $\boldsymbol{\sigma}$ of a physical quantity is determined by the values $\boldsymbol{g}(s)$ of a second quantity at all times s in the past. In (1.1), $\boldsymbol{g}(s)$ is the value of the second quantity s time units ago. If the functional $\mathfrak{F}$ is given, the relation (1.1) may be used to predict the present value $\boldsymbol{\sigma}$ of the first quantity from a knowledge of the "history" $\boldsymbol{g}$ of the second quantity. The relation (1.1) may be interpreted as expressing the causal nature of a class of physical processes. For definiteness we assume that the values of $\boldsymbol{\sigma}$ and $\boldsymbol{g}(s)$ lie in appropriate normed vector spaces, not necessarily finite-dimensional.

In many physical situations, the value $\boldsymbol{\sigma}$ in (1.1) will be, in some sense, more sensitive to the values of $\boldsymbol{g}$ for small s than for large s. Intuitively speaking, the "memory" of the system described by (1.1) will "fade away" in time. In order to make this idea precise, we shall introduce a *norm* $\|\boldsymbol{g}\|$ in the function space of the histories $\boldsymbol{g}$ for which (1.1) is meaningful. We first choose a number p,

$1 \leqq p \leqq \infty$, which will be kept fixed, and then define the norm of $\boldsymbol{g}$ to be

$$\|\boldsymbol{g}\| = \begin{cases} \sqrt[p]{\int\limits_0^\infty (|\boldsymbol{g}(s)|\, h(s))^p\, ds} & \text{if } \ 1 \leqq p < \infty \\ \sup\limits_{s \geqq 0} |\boldsymbol{g}(s)|\, h(s) & \text{if } \ p = \infty, \end{cases} \tag{1.2}$$

where $|\boldsymbol{g}(s)|$ is the magnitude of $\boldsymbol{g}(s)$ and where $h(s)$ is a real-valued function which approaches zero rapidly as $s \to \infty$. Thus, in computing the norm $\|\boldsymbol{g}\|$ we assign a greater influence to the values of $\boldsymbol{g}$ for small s (recent past) than for very large s (distant past). We call the function h in (1.2) an *influence function*. The physical idea that the memory of the system is fading corresponds to the mathematical assumption that the functional $\mathfrak{F}$ in (1.1) is *continuous* with respect to convergence in the norm (1.2) of the function space of histories. The influence function h characterizes the rapidity with which the memory is fading.

It is always possible to reduce the relation (1.1) to one which is *normalized* in the sense that the possible histories $\boldsymbol{g}$ have the value $\boldsymbol{g}(0) = \boldsymbol{0}$ at the present instant and that the value of $\mathfrak{F}$ for the zero history $\boldsymbol{g}(s) \equiv \boldsymbol{0}$ is zero. Now, physical experience indicates that phenomena which one would expect to be described exactly by a normalized functional relation (1.1) often follow a simpler relation of the form

$$\boldsymbol{\sigma} = \mathfrak{l}(\overset{(1)}{\boldsymbol{g}}), \quad \overset{(1)}{\boldsymbol{g}} = \frac{d}{ds} \boldsymbol{g}(s)\Big|_{s=0}, \tag{1.3}$$

where $\mathfrak{l}$ is a linear transformation. In particular, it appears that (1.3) accounts well for the observed phenomena in the case of slow processes. This observation leads to the conjecture that, in some mathematically precise sense, the relation (1.3) approximates the normalized relation (1.1) for slow processes. It is the purpose of the present paper to prove an approximation theorem of this kind. Theorem 2 of §5 asserts not only that (1.3) is the first-order approximation to the normalized relation (1.1) for slow processes but shows also the form of the approximations of higher order. The theorem is based on the assumption that the functional $\mathfrak{F}$ of (1.1) is not only continuous but also Fréchet differentiable at the zero history $\boldsymbol{g}(0) \equiv \boldsymbol{0}$ in the function space with norm (1.2). A normalized functional $\mathfrak{F}$ satisfying this assumption will be called a *memory functional*. The continuity of $\mathfrak{F}$ for histories $\boldsymbol{g}$ different from the zero history is not needed to prove the approximation theorem.

In Part II of the paper we apply the approximation theorem to constitutive equations of continuum mechanics. Our main interest there is in the logical status of the theory of Newtonian fluids within the framework of a recently proposed general theory of fluids with memory effects*.

The theory of compressible Newtonian fluids is based on the following constitutive equation for the stress tensor $\boldsymbol{S}$:

$$\boldsymbol{S} = -p\,\boldsymbol{I} + 2\eta\,\boldsymbol{D} + \lambda(\operatorname{tr} \boldsymbol{D})\,\boldsymbol{I}; \tag{1.4}$$

here the rate of deformation tensor $\boldsymbol{D}$ is the symmetric part of the velocity gradient tensor; $p = p(\varrho)$ is the hydrostatic pressure the fluid would be supporting if it were at rest at its present density ϱ; η and λ are functions of ϱ alone and are called coefficients of viscosity.

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* *Cf.* [*1*], [*2*] and [*3*].

Experience shows that for some substances, such as water, the theory of Newtonian fluids accounts remarkably well for experimental measurements over a very wide range of conditions. Other substances, such as molten plastics, definitely do not obey (1.4) exactly, but yet their behavior appears to approximate that of Newtonian fluids in the limit of slow motions.

In §6 we review the mathematical definition of a *simple fluid*. As we have frequently remarked in the past, we believe this definition is capable of covering the behavior of nearly all real fluids including such substances as molten plastics which exhibit "hereditary", "non-Newtonian" and "viscoelastic" effects. Here we add to the definition of a simple fluid a new requirement: we require that the functionals occurring in the definition be memory functionals in the sense of the definition used in this paper. As we have indicated, this requirement is related to the physical assumption that simple fluids have a fading memory. If this assumption were not satisfied, the term "fluid" would hardly be appropriate.

In §7 we conclude that the theory of Newtonian fluids is indeed the complete first-order approximation to the theory of simple fluids for slow flows. We also indicate what an experimenter should expect to find as second-order corrections to the constitutive equation of a Newtonian fluid. We point out that several special flow problems for incompressible second-order fluids lead to third-order linear partial differential equations.

In §8 we apply our approximation theorem to the theory of the general simple materials defined in [*1*], Part III.

I. The Approximation Theorem

2. Influence Functions and Histories

An *influence function* h of order $r>0$ is a real-valued function of a real variable with the following properties:

(i) $h(s)$ is defined and continuous for $0 \leqq s < \infty$.

(ii) $h(s)$ is positive, $h(s) > 0$.

(iii) For each $\sigma > 0$, there is a constant M_σ, independent of α, such that

$$\sup_{s \geqq \sigma} \frac{h(s/\alpha)}{\alpha^r h(s)} \leqq M_\sigma \quad \text{for} \quad 0 < \alpha \leqq 1 . \tag{2.1}$$

The last condition (iii) means that $h(s)$ must decay to zero at a fast enough rate as $s \to \infty$. In fact, we have

$$\sup_{s>0} s^r h(s) = N < \infty , \tag{2.2}$$

which follows from (2.1) by taking $\sigma = 1$ and observing that

$$\left(\frac{1}{\alpha}\right)^r h\left(\frac{1}{\alpha}\right) \leqq h(1) \sup_{s \geqq 1} \frac{h(s/\alpha)}{\alpha^r h(s)} \leqq h(1)\, M_1 .$$

Let a real Banach space $\mathcal{S}$ with norm $|\;|$ be given. We then define a *history* $\boldsymbol{g}$ to be a measurable function defined for $0 \leqq s < \infty$ with values $\boldsymbol{g}(s)$ in $\mathcal{S}$. Two such functions will be regarded as the same if they differ only on a set of measure zero.

For a given influence function h and a given number p, $1 \leqq p \leqq \infty$, we define the *$\mathcal{L}_{h,p}$-norm*, $\|\boldsymbol{g}\|_{h,p}$, or simply $\|\boldsymbol{g}\|$, of a history $\boldsymbol{g}$ by

$$\|\boldsymbol{g}\|_{h,p} = \sqrt[p]{\int_0^\infty (|\boldsymbol{g}(s)|\, h(s))^p\, ds} \quad \text{if} \quad 1 \leqq p < \infty, \tag{2.3}$$

$$\|\boldsymbol{g}\|_{h,\infty} = \sup_{s \geqq 0} |\boldsymbol{g}(s)|\, h(s) \qquad \text{if} \quad p = \infty. \tag{2.4}$$

In (2.4) and subsequently *sup* $f(s)$ stands for the essential supremum of $f(s)$, *i.e.* for the greatest lower bound of the suprema of all functions which differ from $f(s)$ only on a set of measure zero.

The set of all histories with finite $\mathcal{L}_{h,p}$-norm forms a Banach space, which we denote by $\mathcal{L}_{h,p}$.

We remark that (2.2) is a necessary but not a sufficient condition for the decay relation (2.1). However, if a function $h(s)$ satisfies (i), (ii), and the limit relation

$$\lim_{s \to \infty} s^r h(s) = 0 \quad \text{monotonically for large } s, \tag{2.5}$$

then it also satisfies the decay relation (2.1) and hence is an influence function of order r. To prove this fact we consider the expression

$$\frac{h(s/\alpha)}{\alpha^r h(s)} = \frac{(s/\alpha)^r h(s/\alpha)}{s^r h(s)}, \quad 0 < \alpha \leqq 1. \tag{2.6}$$

Since $s^r h(s)$ is monotonically decreasing for s larger than some value s_1, it follows that (2.6) is not greater than 1 for all $s > s_1$. For $0 < \sigma \leqq s \leqq s_1$ the denominator $s^r h(s)$ of (2.6) has a positive minimum because h satisfies the conditions (i) and (ii). The numerator of (2.6) is bounded by the maximum of $s^r h(s)$, which exists and is finite because of (2.5). It follows that (2.6) is bounded, for $\sigma \leqq s < \infty$, by a constant M_σ independent of α, which is the content of (2.1).

The function

$$h(s) = \frac{1}{(s+1)^r}$$

is an influence function of order r. An exponential,

$$h(s) = e^{-\beta s}, \qquad \beta > 0,$$

is an influence function of any order.

3. Retardation

The *retardation Γ_α* with *retardation factor* α, $0 < \alpha \leqq 1$, is the linear transformation $\boldsymbol{g} \to \boldsymbol{g}_\alpha$ defined, for all histories $\boldsymbol{g}$, by

$$(\Gamma_\alpha \boldsymbol{g})(s) = \boldsymbol{g}_\alpha(s) = \boldsymbol{g}(\alpha s). \tag{3.1}$$

We show that Γ_α maps the space $\mathcal{L}_{h,p}$ into itself. For $p \neq \infty$, we have*

$$\begin{aligned} \|\boldsymbol{g}_\alpha\|^p &= \int_0^\infty (|\boldsymbol{g}(\alpha s)|\, h(s))^p\, ds \\ &= \frac{1}{\alpha} \int_0^\infty (|\boldsymbol{g}(s)|\, h(s))^p \left(\frac{h(s/\alpha)}{h(s)}\right)^p ds \leqq \|\boldsymbol{g}\|^p \frac{1}{\alpha} \left(\sup_{s>0} \frac{h(s/\alpha)}{h(s)}\right)^p. \end{aligned} \tag{3.2}$$

---

* Since no confusion can arise, we omit the indices h, p on the norm.

Since h is continuous and positive,

$$\sup_{\sigma \geqq s \geqq 0} \frac{h(s/\alpha)}{h(s)} < \infty$$

for any $\sigma > 0$. The decay condition (2.1) implies that

$$\sup_{s \geqq \sigma} \frac{h(s/\alpha)}{h(s)} \leqq \alpha^r M_\sigma < \infty .$$

Hence,

$$\sup_{s \geqq 0} \frac{h(s/\alpha)}{h(s)} = K_\alpha < \infty . \tag{3.3}$$

Combining (3.2) and (3.3) gives

$$\|\boldsymbol{g}_\alpha\| \leqq \alpha^{-\frac{1}{p}} K_\alpha \|\boldsymbol{g}\| . \tag{3.4}$$

It is easily verified that this inequality remains valid for $p = \infty$ when we put $\sqrt[\infty]{\alpha} = 1$. It follows from (3.4) that when $\boldsymbol{g}$ has a finite norm $\boldsymbol{g}_\alpha$ has a finite norm and is, therefore, in $\mathscr{L}_{h,p}$. Intuitively, retardation replaces a given history by one which is essentially the same, but slower.

If, possibly after a suitable alteration of $\boldsymbol{g}(s)$ on a set of measure zero, the limits

$$\overset{(0)}{\boldsymbol{g}} = \lim_{s \to 0} \boldsymbol{g}(s) , \qquad \overset{(h)}{\boldsymbol{g}} = \lim_{s \to 0} \frac{k!}{s^k} \left(\boldsymbol{g}(s) - \sum_{j=0}^{k-1} \frac{s^j}{j!} \overset{(j)}{\boldsymbol{g}} \right) \tag{3.5}$$

exist for $k = 0, 1, \ldots, n$, then we shall say that the history $\boldsymbol{g}$ has n *generalized derivatives* at $s = 0$. Of course, the existence of ordinary derivatives implies the existence of the corresponding generalized derivatives. Here we shall use the term "derivative" always in the generalized sense of (3.5). Generalized derivatives at $s = 0$ of any order may exist even though $\boldsymbol{g}(s)$ is not continuous near $s = 0$.

The *Taylor transformation* Π_n is the linear transformation $\boldsymbol{g} \to \Pi_n \boldsymbol{g}$ defined, for all histories $\boldsymbol{g}$ which are n times differentiable at $s = 0$, by

$$(\Pi_n \boldsymbol{g})(s) = \sum_{j=0}^{n} \frac{s^j}{j!} \overset{(j)}{\boldsymbol{g}} , \tag{3.6}$$

where the $\overset{(j)}{\boldsymbol{g}}$ are the derivatives (3.5).

This Taylor transformation replaces $\boldsymbol{g}$ by its n^{th} "Taylor approximation" $\Pi_n \boldsymbol{g}$. The history $\Pi_n \boldsymbol{g}$ is a polynomial of degree $\leqq n$.

We note that retardation Γ_α preserves the differentiability of a history and that Π_n and Γ_α commute; *i.e.*, for all histories $\boldsymbol{g}$ having n derivatives at $s = 0$, we have

$$\Pi_n \Gamma_\alpha \boldsymbol{g} = \Gamma_\alpha \Pi_n \boldsymbol{g} . \tag{3.7}$$

The set of all histories which have n derivatives at $s = 0$ and which also belong to the Banach space $\mathscr{L}_{h,p}$ will be denoted by $\mathscr{D}_n$. This set $\mathscr{D}_n$ is a linear subspace of $\mathscr{L}_{h,p}$, but it is not closed in $\mathscr{L}_{h,p}$ and hence not a Banach space.

The following theorem is an analogue of the classical Taylor approximation theorem.

Theorem 1. *Assume that n, p, and the order r of the influence function h satisfy the inequality*

$$n < r - \frac{1}{p} \qquad \left(\frac{1}{p} = 0 \quad \text{if} \quad p = \infty \right) . \tag{3.8}$$

Then the Taylor transformation Π_n maps the subspace $\mathcal{D}_n$ of $\mathcal{L}_{h,p}$ into itself, and, for all $\boldsymbol{g}$ in $\mathcal{D}_n$,

$$\lim_{\alpha\to 0} \frac{1}{\alpha^n} \|\Gamma_\alpha(\boldsymbol{g}-\Pi_n \boldsymbol{g})\|_{h,p} = 0. \tag{3.9}$$

We can also write (3.9), using (3.7), in the form

$$\Gamma_\alpha \boldsymbol{g} - \Pi_n \Gamma_\alpha \boldsymbol{g} = \boldsymbol{g}_\alpha - \Pi_n \boldsymbol{g}_\alpha = o(\alpha^n), \tag{3.10}$$

where the order symbol o must be understood in terms of the norm of the function space $\mathcal{L}_{h,p}$. Roughly speaking, the theorem states that in the space $\mathcal{D}_n \in \mathcal{L}_{h,p}$ a slow history is close to its Taylor approximation and that the distance between them is $o(\alpha^n)$.

Proof of Theorem 1. We consider only the case when p is finite. The case when $p=\infty$ can be treated analogously.

First, we show that every polynomial of degree $\leqq n$ has a finite norm. For this purpose it is sufficient to prove that

$$\int_0^\infty (s^k h(s))^p \, ds = L_k < \infty \quad \text{for} \quad k \leqq n. \tag{3.11}$$

It follows from (2.2) that

$$\int_1^\infty (s^k h(s))^p \, ds = \int_1^\infty (s^r h(s))^p s^{p(k-r)} \, ds \leqq N^p \int_1^\infty s^{-p(r-k)} \, ds. \tag{3.12}$$

The inequalities (3.8) and $k \leqq n$ imply that $p(r-k) > 1$ and hence that the integral

$$\int_1^\infty s^{-p(r-k)} \, ds = \frac{1}{p(r-k)-1}$$

is finite. Since $s^k h(s)$ is continuous, it then follows from (3.12) that the integral (3.11) is finite.

We have shown that any polynomial of degree $\leqq n$ belongs to $\mathcal{L}_{h,p}$ and hence to $\mathcal{D}_n$, because it has n derivatives at $s=0$. Since a Taylor approximation $\Pi_n \boldsymbol{g}$ is a polynomial of degree $\leqq n$, it follows that Π_n maps $\mathcal{D}_n$ into itself.

The definitions (3.5) and (3.6) imply that the history

$$\boldsymbol{f} = \boldsymbol{g} - \Pi_n \boldsymbol{g} \in \mathcal{D}_n \tag{3.13}$$

satisfies the limit relation

$$\lim_{s\to 0} \frac{|\boldsymbol{f}(s)|}{s^n} = 0. \tag{3.14}$$

The definitions (2.3), (3.1), and (3.13) show that the assertion (3.9) of the theorem is equivalent to

$$\lim_{\alpha\to 0} \frac{1}{\alpha^{np}} \int_0^\infty (|\boldsymbol{f}(\alpha s)| \, h(s))^p \, ds = 0. \tag{3.15}$$

We observe that

$$\frac{1}{\alpha^{np}} \int_0^\infty (|\boldsymbol{f}(\alpha s)| \, h(s))^p \, ds = \frac{1}{\alpha^{np+1}} \int_0^\infty \left(|\boldsymbol{f}(s)| \, h\left(\frac{s}{\alpha}\right)\right)^p ds \tag{3.16}$$

and investigate the latter expression. Let $\varepsilon > 0$ be given. It follows from (3.14) that we can find a $\sigma(\varepsilon) > 0$ such that

$$\frac{|\boldsymbol{f}(s)|}{s^n} \leqq \varepsilon \quad \text{for} \quad 0 < s \leqq \sigma(\varepsilon).$$

Hence,

$$\begin{aligned} \frac{1}{\alpha^{np+1}} \int_0^{\sigma(\varepsilon)} \left(|\boldsymbol{f}(s)|\, h\left(\frac{s}{\alpha}\right) \right)^p ds &\leqq \frac{1}{\alpha^{np+1}}\, \varepsilon^p \int_0^{\sigma(\varepsilon)} \left(s^n h\left(\frac{s}{\alpha}\right) \right)^p ds \\ &\leqq \varepsilon^p \int_0^{\infty} \left(\frac{s^n}{\alpha^n} h\left(\frac{s}{\alpha}\right) \right)^p d\left(\frac{s}{\alpha}\right) = \varepsilon^p L_n, \end{aligned} \tag{3.17}$$

where L_n is given by (3.11). On the other hand, we have

$$\frac{1}{\alpha^{np+1}} \int_{\sigma(\varepsilon)}^{\infty} \left(|\boldsymbol{f}(s)|\, h\left(\frac{s}{\alpha}\right) \right)^p ds = \alpha^w \int_{\sigma(\varepsilon)}^{\infty} \left(|\boldsymbol{f}(s)|\, h(s) \right)^p \left(\frac{h(s/\alpha)}{\alpha^r\, h(s)} \right)^p ds, \tag{3.18}$$

where $w = p(r-n) - 1$. Applying the decay condition (2.1) to (3.18), we find that

$$\frac{1}{\alpha^{np+1}} \int_{\sigma(\varepsilon)}^{\infty} \left(|\boldsymbol{f}(s)|\, h\left(\frac{s}{\alpha}\right) \right)^p ds \leqq \alpha^w M^p_{\sigma(\varepsilon)} \|\boldsymbol{f}\|^p. \tag{3.19}$$

On combining (3.17) and (3.19), we see that

$$\frac{1}{\alpha^{np+1}} \int_0^{\infty} \left(|\boldsymbol{f}(s)|\, h\left(\frac{s}{\alpha}\right) \right)^p ds \leqq \varepsilon^p L_n + \alpha^w M^p_{\sigma(\varepsilon)} \|\boldsymbol{f}\|^p \tag{3.20}$$

holds for all $\varepsilon > 0$ and all $0 < \alpha \leqq 1$. The assumed inequality (3.8) insures that $w = p(r-n) - 1$ is positive. Therefore, by choosing first ε and then α sufficiently small, we see that the right side of (3.20) can be made as small as desired. It then follows from (3.16) that the limit relation (3.15) holds, which completes the proof of the theorem.

In the special case $n = 0$, $r > \frac{1}{p}$, $\overset{(0)}{\boldsymbol{g}} = \boldsymbol{0}$ Theorem 1 states the following. If $\boldsymbol{g} \in \mathscr{L}_{h,p}$ is continuous at $s = 0$ with value $\boldsymbol{0}$, then

$$\lim_{\alpha \to 0} \|\Gamma_\alpha \boldsymbol{g}\| = 0. \tag{3.21}$$

If $1 \leqq n < r - \frac{1}{p}$ and $\overset{(0)}{\boldsymbol{g}} = \boldsymbol{0}$, instead of (3.21), we have the stronger result

$$\boldsymbol{g}_\alpha = \Gamma_\alpha \boldsymbol{g} = O(\alpha). \tag{3.22}$$

4. *Memory Functionals*

We first recall some definitions* from the theory of functions defined on a real vector space $\mathscr{H}$ with norm $\| \ \|$ and having values in another real vector space $\mathscr{T}$ with norm $| \ |$.

(1) Suppose $\boldsymbol{g}_1, \ldots, \boldsymbol{g}_k$ are variables in $\mathscr{H}$. Then a function $\mathfrak{P}(\boldsymbol{g}_1, \ldots, \boldsymbol{g}_k)$, defined for all values of the variables $\boldsymbol{g}_i$ in $\mathscr{H}$ and having values in $\mathscr{T}$, is called

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* These definitions are analogous to those given in Chap. XXVI of [*4*].

a *bounded k-linear form* if it is linear in each variable $\boldsymbol{g}_i$ separately and if there is a constant M, independent of $\boldsymbol{g}_i$, such that

$$|\mathfrak{P}(\boldsymbol{g}_1, \ldots, \boldsymbol{g}_k)| \leqq M\,\|\boldsymbol{g}_1\| \cdots \|\boldsymbol{g}_k\|. \tag{4.1}$$

The form $\mathfrak{P}(\boldsymbol{g}_1, \ldots, \boldsymbol{g}_k)$ is said to be *symmetric* if any permutation of the variables leaves the value unchanged.

(2) A function $\mathfrak{P}(\boldsymbol{g})$, defined for all $\boldsymbol{g} \in \mathscr{H}$ and having values in $\mathscr{T}$, is called a *bounded homogeneous polynomial of degree k* if there is a bounded symmetric k-linear form $\mathfrak{P}(\boldsymbol{g}_1, \ldots, \boldsymbol{g}_k)$ such that

$$\mathfrak{P}(\boldsymbol{g}) = \mathfrak{P}(\boldsymbol{g}, \ldots, \boldsymbol{g}). \tag{4.2}$$

The symmetric n-linear form $\mathfrak{P}(\boldsymbol{g}_1, \ldots, \boldsymbol{g}_k)$ is uniquely determined by the homogeneous polynomial $\mathfrak{P}(\boldsymbol{g})$ and is called the *polar form* of the polynomial.

(3) A function $\mathfrak{F}$, defined on a neighborhood of $\boldsymbol{0} \in \mathscr{H}$ and having values in $\mathscr{T}$ is said to be *n times Fréchet-differentiable* at $\boldsymbol{0} \in \mathscr{H}$ if there are bounded homogeneous polynomials $\delta^k \mathfrak{F}(\boldsymbol{g})$ of degree $k = 0, 1, \ldots, n$ such that

$$\mathfrak{F}(\boldsymbol{g}) = \sum_{k=0}^{n} \frac{1}{k!}\, \delta^k \mathfrak{F}(\boldsymbol{g}) + \|\boldsymbol{g}\|^n\, \mathfrak{R}(\boldsymbol{g}), \tag{4.3}$$

where

$$\lim_{\|\boldsymbol{g}\| \to 0} |\mathfrak{R}(\boldsymbol{g})| = 0. \tag{4.4}$$

The polynomial $\delta^k \mathfrak{F}(\boldsymbol{g})$ or its polar form $\delta^k \mathfrak{F}(\boldsymbol{g}_1, \ldots, \boldsymbol{g}_k)$ is called the k^{th} *Fréchet-differential* or the k^{th} *variation* of $\mathfrak{F}$ at $\boldsymbol{0} \in \mathscr{H}$. The differentials may be obtained recursively by

$$\delta^0 \mathfrak{F}(\boldsymbol{g}) = \mathfrak{F}(\boldsymbol{0}), \quad \delta^k \mathfrak{F}(\boldsymbol{g}) = k! \lim_{\lambda \to 0} \frac{1}{\lambda^k} \left[\mathfrak{F}(\lambda \boldsymbol{g}) - \sum_{j=0}^{k-1} \frac{\lambda^j}{j!}\, \delta^j \mathfrak{F}(\boldsymbol{g})\right]. \tag{4.5}$$

We here consider the case in which $\mathscr{H}$ is the function space of all histories $\boldsymbol{g}$ with the following properties:

(α) $\boldsymbol{g}$ has a finite $\mathscr{L}_{h,p}$-norm,

(β) $\boldsymbol{g}$ has n generalized derivatives at $s = 0$,

(γ) $\boldsymbol{g}$ has a zero limit at $s = 0$:

$$\lim_{s \to 0} \boldsymbol{g}(s) = \overset{(0)}{\boldsymbol{g}} = \boldsymbol{0}, \tag{4.6}$$

(δ) n, p, and the order r of the influence function h obey the inequality (3.8).

The conditions (α) and (β) state that $\mathscr{H} \subset \mathscr{D}_n$. Condition ($\delta$) insures that the conclusion (3.9) of Theorem 1 is valid for all $\boldsymbol{g} \in \mathscr{H}$.

A function $\mathfrak{F}$ defined on a neighborhood in $\mathscr{H}$ of the zero function $\boldsymbol{0} \in \mathscr{H} \subset \mathscr{L}_{h,p}$ and having values in a real Banach space $\mathscr{T}$ will be called a *memory functional of type* (h, n) if it is n times Fréchet-differentiable at $\boldsymbol{0} \in \mathscr{H}$ and if it is normalized by

$$\delta^0 \mathfrak{F}(\boldsymbol{g}) = \mathfrak{F}(\boldsymbol{0}) = \boldsymbol{0}. \tag{4.7}$$

In some applications it may be more natural to assume that $\mathfrak{F}$ is defined and Fréchet-differentiable on a neighborhood of zero in the entire space $\mathscr{L}_{h,p}$. However, the approximation theorem of the following section applies only to histories which belong to the subspace $\mathscr{H}$ of $\mathscr{L}_{h,p}$.

5. The Approximation Theorem

Theorem 2. *Let $\mathfrak{F}$ be a memory functional of type (h, n); the histories $\boldsymbol{g}$ in the domain space $\mathscr{H}$ of $\mathfrak{F}$ have values in the space $\mathscr{S}$, and $\mathfrak{F}$ itself has values in $\mathscr{T}$. Then, for each k-tuple of indices $(j_1, j_2, \ldots, j_k)$ such that*

$$1 \leqq j_1 \leqq j_2 \leqq \cdots \leqq j_k \leqq n, \quad j_1 + j_2 + \cdots + j_n \leqq n, \tag{5.1}$$

there exists a bounded k-linear form $\mathfrak{l}_{j_1 \ldots j_k}$ with variables in $\mathscr{S}$ and values in $\mathscr{T}$ such that, for all $\boldsymbol{g} \in \mathscr{H}$,

$$\mathfrak{F}(\boldsymbol{g}_\alpha) = \sum_{(j_1, \ldots, j_k)} \mathfrak{l}_{j_1 \ldots j_k}(\overset{(j_1)}{\boldsymbol{g}}_\alpha, \ldots, \overset{(j_k)}{\boldsymbol{g}}_\alpha) + o(\alpha^n), \tag{5.2}$$

where

$$\boldsymbol{g} = \Gamma_\alpha \boldsymbol{g}, \quad \text{i.e.,} \quad \boldsymbol{g}_\alpha(s) = \boldsymbol{g}(\alpha s), \tag{5.3}$$

and where

$$\overset{(j)}{\boldsymbol{g}}_\alpha = \alpha^j \overset{(j)}{\boldsymbol{g}} \tag{5.4}$$

is the generalized j^{th} derivative at $s=0$ of $\boldsymbol{g}_\alpha$, defined according to (3.5). The sum in (5.2) is to be extended over all sets $(j_1, \ldots, j_k)$ of indices satisfying (5.1), and the order symbol $o(\alpha^n)$ has the usual sense. The multilinear forms $\mathfrak{l}_{j_1 \ldots j_k}$ are uniquely determined by $\mathfrak{F}$.

This theorem permits the asymptotic approximation of a memory *functional*, for "slow" histories, by a polynomial *function* of the derivatives at $s=0$ of the argument function of the functional. It is worth noting that the approximating expression

$$\mathfrak{F}_n(\boldsymbol{g}) = \sum_{(j_1, \ldots, j_k)} \mathfrak{l}_{j_1 \ldots j_k}(\overset{(j_1)}{\boldsymbol{g}}, \ldots, \overset{(j_k)}{\boldsymbol{g}}), \tag{5.6}$$

regarded as a functional of $\boldsymbol{g}$, is not a memory functional in the sense of the previous section because it is not even continuous at the zero function $\boldsymbol{g} = \boldsymbol{0} \in \mathscr{H}$.

For $n=1$ and $n=2$, (5.6) reduces to

$$\mathfrak{F}_1(\boldsymbol{g}) = \mathfrak{l}_1(\overset{(1)}{\boldsymbol{g}}), \quad \mathfrak{F}_2(\boldsymbol{g}) = \mathfrak{l}_1(\overset{(1)}{\boldsymbol{g}}) + \mathfrak{l}_{11}(\overset{(1)}{\boldsymbol{g}}, \overset{(1)}{\boldsymbol{g}}) + \mathfrak{l}_2(\overset{(2)}{\boldsymbol{g}}) \tag{5.7}$$

respectively, where $\mathfrak{l}_1$ and $\mathfrak{l}_2$ are linear and $\mathfrak{l}_{11}$ is bilinear.

Proof of Theorem 2. For $n=0$, the theorem is a trivial consequence of (3.21). For $n \geqq 1$, the proof is based on a combination of the Fréchet-differentiability assumption (4.3) and equation (3.10) of Theorem 1 which, in the notation of (5.3), may be written as

$$\boldsymbol{g}_\alpha = \Pi_n \boldsymbol{g}_\alpha + o(\alpha^n). \tag{5.8}$$

Since $\overset{(0)}{\boldsymbol{g}} = \boldsymbol{0}$ by (4.6), the definition (3.6) gives

$$(\Pi_n \boldsymbol{g}_\alpha)(s) = \sum_{j=1}^{n} \alpha^j \frac{s^j}{j!} \overset{(j)}{\boldsymbol{g}}. \tag{5.9}$$

The result (3.22) applies to both $\boldsymbol{g} \in \mathscr{H}$ and $\Pi_n \boldsymbol{g} \in \mathscr{H}$:

$$\Pi_n \boldsymbol{g}_\alpha = O(\alpha), \tag{5.10}$$

$$\boldsymbol{g}_\alpha = O(\alpha). \tag{5.11}$$

Combining (5.11) with (4.3), (4.4), and (4.7), we obtain

$$\mathfrak{F}(\boldsymbol{g}_\alpha) = \sum_{k=1}^{n} \frac{1}{k!}\, \delta^k \mathfrak{F}(\boldsymbol{g}_\alpha) + o(\alpha^n). \tag{5.12}$$

Consider now a bounded homogeneous polynomial $\mathfrak{P}(\boldsymbol{g})$ of degree k, $1 \leqq k \leqq n$, in the sense of (2) of §4. The differentials $\delta^k \mathfrak{F}$, $k > 1$ are such polynomials. The polar form of $\mathfrak{P}$ will also be denoted by $\mathfrak{P}$. Using the multilinearity of this polar form $\mathfrak{P}$, we obtain by (4.2), (5.8), and (5.10)

$$\begin{aligned}\mathfrak{P}(\boldsymbol{g}_\alpha) &= \mathfrak{P}\big(\Pi_n \boldsymbol{g}_\alpha + o(\alpha^n), \ldots, \Pi_n \boldsymbol{g}_\alpha + o(\alpha^n)\big)\\ &= \mathfrak{P}(\Pi_n \boldsymbol{g}_\alpha, \ldots, \Pi_n \boldsymbol{g}_\alpha) + \sum \mathfrak{P}\big(O(\alpha), \ldots, o(\alpha^n), \ldots, O(\alpha)\big),\end{aligned}$$

where each term of the sum contains at least one variable $o(\alpha^n)$. It follows from the boundedness (4.1) of $\mathfrak{P}$ that the terms in the sum are all $o(\alpha^{n+k-1})$. Since $k \geqq 1$, we have

$$\mathfrak{P}(\boldsymbol{g}_\alpha) = \mathfrak{P}(\Pi_n \boldsymbol{g}_\alpha) + o(\alpha^n). \tag{5.13}$$

We now investigate $\mathfrak{P}(\Pi_n \boldsymbol{g}_\alpha)$. Using (5.9) and the multilinearity of the polar form $\mathfrak{P}$, we obtain

$$\mathfrak{P}(\Pi_n \boldsymbol{g}_\alpha) = \sum_{j_1=1}^{n} \cdots \sum_{j_k=1}^{n} \frac{\alpha^{j_1 + \cdots + j_k}}{j_1! \ldots j_k!}\, \mathfrak{P}\big(s^{j_1} \overset{(j_2)}{\boldsymbol{g}}, \ldots, s^{j_k} \overset{(j_k)}{\boldsymbol{g}}\big). \tag{5.14}$$

Due to the symmetry of $\mathfrak{P}$, all terms of (5.14) which differ only in the order of the indices $j_1, \ldots, j_k$ are the same. Collecting these terms and separating all terms of order higher than n in α, we get

$$\mathfrak{P}(\Pi_n \boldsymbol{g}_\alpha) = \sum_{\substack{(j_1, \ldots, j_k)\\ k \text{ fixed}}} \alpha^{j_1 + \cdots + j_k}\, m_{j_1 \ldots j_k}\, \mathfrak{P}\big(s^{j_1} \overset{(j_1)}{\boldsymbol{g}}, \ldots, s^{j_k} \overset{(j_k)}{\boldsymbol{g}}\big) + O(\alpha^{n+1}) \tag{5.15}$$

where the sum is to be extended over all k-tuples $(j_1, \ldots, j_k)$ satisfying (5.1) and where the $m_{j_1 \ldots j_k}$ are positive rational numbers.

Now, the function

$$\mathfrak{p}_{j_1 \ldots j_k}(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k) = \mathfrak{P}(s^{j_1} \boldsymbol{a}_1, \ldots, s^{j_k} \boldsymbol{a}_k), \tag{5.16}$$

with variables $\boldsymbol{a}_i \in \mathscr{S}$ and values in $\mathscr{T}$ is clearly multilinear, because $\mathfrak{P}$ is. Also, $\mathfrak{p}_{j_1 \ldots j_k}$ is bounded. Indeed, application of (4.1) to (5.16) yields

$$|\mathfrak{p}_{j_1 \ldots j_k}(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k)| \leqq M \, \| s^{j_1} \boldsymbol{a}_1 \| \ldots \| s^{j_k} \boldsymbol{a}_k \|,$$

and (3.11) shows

$$\| s^{j_l} \boldsymbol{a}_l \|_h = |\boldsymbol{a}_l| \sqrt[p]{L_{j_l}},$$

hence

$$|\mathfrak{p}_{j_1 \ldots j_k}(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k)| \leqq \bar{M} \, |\boldsymbol{a}_1| \ldots |\boldsymbol{a}_k|. \tag{5.17}$$

Substituting (5.16) into (5.15) and using (5.4) yields

$$\mathfrak{P}(\Pi_n \boldsymbol{g}_\alpha) = \sum_{\substack{(j_1, \ldots, j_k)\\ k \text{ fixed}}} \mathfrak{l}_{j_1 \ldots j_k}\big(\overset{(j_1)}{\boldsymbol{g}}_\alpha, \ldots, \overset{(j_k)}{\boldsymbol{g}}_\alpha\big) + O(\alpha^{n+1}) \tag{5.18}$$

where the $\mathfrak{l}_{j_1 \ldots j_k}$ are bounded k-linear forms.

Applying the results (5.13) and (5.18) to $\mathfrak{P} = \frac{1}{k!}\,\delta^k \mathfrak{F}$ in (5.12), we obtain formula (5.2).

The uniqueness of the $\mathfrak{l}_{j_1 \dots j_k}$ follows easily from the observation that they are linearly independent and of order $O(\alpha^n)$ or lower.

We remark that the multilinear forms $\mathfrak{l}_{j_1 \dots j_k}$ of (5.2) are not necessarily symmetric.

II. Applications

6. The Concept of a Simple Fluid

The notion of a simple fluid has been given a definition within the framework of a general theory of the mechanical behavior of materials*. This definition is based on the following two physical assumptions**:

(a) The present stress depends on the past history of the first spatial gradient of the displacement function.

(b) A fluid has no preferred configurations.

Using the principle of material objectivity ([*1*], §11), it was shown in reference [*1*] that the constitutive equation of a simple fluid can be written in the form (22.12) of [*1*]. This functional relation, in a slightly different notation, reads

$$\boldsymbol{S}(t) = \mathop{\mathfrak{H}}_{s=0}^{\infty}\big(\boldsymbol{C}_t(t-s)\,;\,\varrho(t)\big). \tag{6.1}$$

Here $\boldsymbol{S}(t)$ is the stress and $\varrho(t)$ the density at time t. $\boldsymbol{C}_t(\tau)$ is called the right Cauchy-Green tensor at time τ relative to the configuration at time t. This tensor is defined by

$$\boldsymbol{C}_t(\tau) = \boldsymbol{F}_t^T(\tau)\,\boldsymbol{F}_t(\tau) \tag{6.2}$$

where

$$\boldsymbol{F}_t(\tau) = \nabla_{\boldsymbol{x}}\,\boldsymbol{\chi}_t(\boldsymbol{x}, \tau) \tag{6.3}$$

is the gradient the displacement function $\boldsymbol{\xi} = \boldsymbol{\chi}_t(\boldsymbol{x}, \tau)$ which gives the position at time τ of the material point having the position $\boldsymbol{x}$ at time t. The stress tensor $\boldsymbol{S}(t)$ is symmetric. The Cauchy-Green tensor $\boldsymbol{C}_t(\tau)$ is positive definite and symmetric; for $\tau = t$ it reduces to the unit tensor $\boldsymbol{I}$:

$$\boldsymbol{C}_t(t) = \boldsymbol{I}. \tag{6.4}$$

The functional $\mathfrak{H}$ in (6.1) is isotropic; *i.e.*, $\mathfrak{H}$ obeys

$$\boldsymbol{Q}\mathop{\mathfrak{H}}_{s=0}^{\infty}\big(\boldsymbol{C}(s)\,;\,\varrho\big)\,\boldsymbol{Q}^T = \mathop{\mathfrak{H}}_{s=0}^{\infty}\big(\boldsymbol{Q}\,\boldsymbol{C}(s)\,\boldsymbol{Q}^T;\,\varrho\big) \tag{6.5}$$

identically in the history $\boldsymbol{C}(s) = \boldsymbol{C}_t(t-s)$ and the orthogonal tensor $\boldsymbol{Q}$; here $\boldsymbol{Q}^T$ is the transpose of $\boldsymbol{Q}$. It follows from (6.5) that the value of $\mathfrak{H}$ for the "rest history" $\boldsymbol{C}(s) \equiv \boldsymbol{I}$ is a scalar multiple $-p(\varrho)$ of the unit tensor $\boldsymbol{I}$. Defining

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* [*1*], § 21.

** In [*3*] we give a survey of the theory of simple fluids with emphasis on physical applications. In that paper we anticipate some of the results rigorously derived here. Although the language and the definitions of the present paper are slightly different from those used in §§ 6 and 7 of [*3*], the arguments presented here prove also the theorems stated there.

$$\underset{s=0}{\overset{\infty}{\mathfrak{F}}}\left(\boldsymbol{G}(s), \varrho\right) = p(\varrho)\, \boldsymbol{I} + \underset{s=0}{\overset{\infty}{\mathfrak{H}}}\left(\boldsymbol{I} + \boldsymbol{G}(s), \varrho\right), \tag{6.6}$$

we may rewrite (6.1) in the form

$$\boldsymbol{S}(t) = -\, p\left(\varrho(t)\right) \boldsymbol{I} + \underset{s=0}{\overset{\infty}{\mathfrak{F}}}\left(\boldsymbol{C}_t(t-s) - \boldsymbol{I}, \varrho(t)\right). \tag{6.7}$$

The functional $\underset{s=0}{\overset{\sim}{\mathfrak{F}}}\left(\boldsymbol{G}(s)\,;\, \varrho\right)$ is defined for functions $\boldsymbol{G}(s)$ with the property

$$\boldsymbol{G}(0) = \boldsymbol{0}, \tag{6.8}$$

and it has the value $\boldsymbol{0}$ for the zero function $\boldsymbol{G}(s) \equiv \boldsymbol{0}$; *i.e.*,

$$\underset{s=0}{\overset{\infty}{\mathfrak{F}}}(\boldsymbol{0}, \varrho) = \boldsymbol{0}. \tag{6.9}$$

It is also isotropic in the sense of (6.5).

We now assume that for each simple fluid defined by a constitutive equation (6.7) there exists an influence function h of an order r such that the functional $\mathfrak{F}$ of (6.7) is a memory functional of type (h, n) in the sense of §4. The domain of $\mathfrak{F}$ is a class $\mathscr{H}$ of histories $\boldsymbol{G}$ whose values $\boldsymbol{G}(s)$ are in the space $\mathscr{S}$ of all symmetric tensors. For the norm $|\;|$ in $\mathscr{S}$ we use

$$|\boldsymbol{A}| = \sqrt{\operatorname{tr} \boldsymbol{A}^2} \quad \text{for} \quad \boldsymbol{A} \in \mathscr{S}. \tag{6.10}$$

The range space $\mathscr{T}$ of $\mathfrak{F}$ is the same as the range space of the histories $\boldsymbol{G}$, *i.e.* the space $\mathscr{T} = \mathscr{S}$ of all symmetric tensors with norm (6.10). Equation (6.9) insures that $\mathfrak{F}$ has the normalization (4.7) required for a memory functional. The density ϱ enters into (6.7) only as a real parameter. The assumption that $\mathfrak{F}$ is a memory functional implies that its domain of definition contains a neighborhood of zero in a function space $\mathscr{H}$ which is defined by the conditions (α)–(δ) of §4. The condition (γ) expresses the assumption that all histories $\boldsymbol{G} \in \mathscr{H}$ correspond to motions which are continuous at the present instant $s = 0$ (*cf.* (6.8)).

If the simple fluid under consideration is *incompressible*, we must make some alterations in our starting assumptions. For every possible motion in such a fluid the density ϱ is constant and the tensor $\boldsymbol{C}_t(\tau)$ is unimodular. In addition, the stress is determined by the history of the motion only up to a hydrostatic pressure p. Consequently, the equation (6.7) must be replaced by

$$\boldsymbol{S}(t) = -\, p\, \boldsymbol{I} + \underset{s=0}{\overset{\infty}{\mathfrak{F}}}\left(\boldsymbol{C}_t(t-s) - \boldsymbol{I}\right) \tag{6.11}$$

in which the indeterminate pressure p and the functional $\mathfrak{F}$ may be normalized by

$$0 = \operatorname{tr} \boldsymbol{S}(t) + 3p = \operatorname{tr} \underset{s=0}{\overset{\infty}{\mathfrak{F}}}\left(\boldsymbol{C}_t(t-s) - \boldsymbol{I}\right). \tag{6.12}$$

If we were to limit the domain of the functional $\mathfrak{F}$ in (6.11) to kinematically possible histories, this domain would not contain a neighborhood of the zero function in an appropriate function space, $\mathscr{H}$; therefore, $\mathfrak{F}$ could not then be a memory functional. We assume, however, that $\mathfrak{F}$ becomes a memory functional when its domain is extended by putting

$$\overset{\infty}{\underset{s=0}{\mathfrak{F}}}\left(\boldsymbol{C}(s)-\boldsymbol{I}\right)=\overset{\infty}{\underset{s=0}{\mathfrak{F}}}\left((\det \boldsymbol{C}(s))^{-1}\boldsymbol{C}(s)-\boldsymbol{I}\right) \tag{6.13}$$

when $\boldsymbol{C}(s)$ is not unimodular.

Aside from the added properties (6.12) and (6.13), the functional $\mathfrak{F}$ occurring in (6.11) is assumed to be of the same type as that in (6.7) with respect to both isotropy and memory.

7. *Approximations of Order n for Simple Fluids*

We now apply Theorem 2, §5, to the memory functional occurring in the constitutive equations for simple fluids (6.7) or (6.11).

Suppose a motion with Cauchy-Green tensor $\boldsymbol{C}_t(\tau)$ is given. The corresponding history $\boldsymbol{G}$ is defined by

$$\boldsymbol{G}(s)=\boldsymbol{C}_t(t-s)-\boldsymbol{I}. \tag{7.1}$$

If $\boldsymbol{C}_t(\tau)$ is n times differentiable with respect to τ at $\tau=t$, the k^{th} Rivlin-Ericksen tensor $\boldsymbol{A}_k$, $k=1, 2, \ldots, n$ is defined as follows:

$$\boldsymbol{A}_k=\frac{d^k}{d\tau^k}\,\boldsymbol{C}_t(\tau)\Big|_{\tau=t}=(-1)^k\,\overset{(k)}{\boldsymbol{G}}, \tag{7.2}$$

where $\overset{(k)}{\boldsymbol{G}}$ is the k^{th} derivative of $\boldsymbol{G}(s)$ at $s=0$, as in §2.

We now consider histories $\boldsymbol{G}_\alpha$ obtained from $\boldsymbol{G}\in\mathscr{H}$ by retardation as in (5.3). The corresponding Rivlin-Ericksen tensors are

$$\boldsymbol{A}_k^\alpha=\alpha^k\,\boldsymbol{A}_k=(-1)^k\,\overset{(k)}{\boldsymbol{G}}_\alpha. \tag{7.3}$$

They differ only by the inessential factor $(-1)^k$ from the tensors $\overset{(k)}{\boldsymbol{G}}_\alpha$ to be substituted for $\overset{(k)}{\boldsymbol{g}}_\alpha$ in the approximation formula (5.2). This formula, applied to the constitutive equations (6.7) or (6.11), yields the following expression for the stress tensor $\boldsymbol{S}_\alpha$ corresponding to the retarded history $\boldsymbol{G}_\alpha$:

$$\boldsymbol{S}_\alpha=-p\,\boldsymbol{I}+\sum_{(j_1,\ldots j_k)}\mathfrak{m}_{j_1\ldots j_k}(\boldsymbol{A}_{j_1}^\alpha,\ldots,\boldsymbol{A}_{j_k}^\alpha)+o(\alpha^n), \tag{7.4}$$

where the summation is to be extended over all sets of indices $(j_1,\ldots,j_k)$ obeying (5.1). The terms $\mathfrak{m}_{j_1\ldots j_k}(\boldsymbol{A}_{j_1}^\alpha,\ldots,\boldsymbol{A}_{j_k}^\alpha)$ are linear in each of the variables. For compressible fluids, it is understood that p and $\mathfrak{m}_{j_1\ldots j_k}$ depend on the density ϱ.

The equation (7.4) remains valid even when the derivatives shown in (7.2) exist only in the generalized sense of (3.5).

The multilinear forms $\mathfrak{m}_{j_1\ldots j_k}$ in (7.4) are isotropic functions, which means that they obey the identities

$$\boldsymbol{Q}\,\mathfrak{m}_{j_1\ldots j_k}(\boldsymbol{U}_1,\ldots,\boldsymbol{U}_k)\,\boldsymbol{Q}^T=\mathfrak{m}_{j_1\ldots j_k}(\boldsymbol{Q}\,\boldsymbol{U}_1\,\boldsymbol{Q}^T,\ldots,\boldsymbol{Q}\,\boldsymbol{U}_k\,\boldsymbol{Q}^T) \tag{7.5}$$

for all orthogonal $\boldsymbol{Q}$ and all symmetric tensors $\boldsymbol{U}_1,\ldots,\boldsymbol{U}_k$. This proposition follows easily from the fact that the memory functional $\mathfrak{F}$ occurring in (6.7) or (6.11) is isotropic in the sense of (6.5), from the observation that the conjugation $\boldsymbol{G}(s)\rightarrow\boldsymbol{Q}\,\boldsymbol{G}(s)\,\boldsymbol{Q}^T$ leaves the norm $\|\boldsymbol{G}\|$ unchanged, and from the uniqueness of the multilinear forms $\mathfrak{m}_{j_1\ldots j_k}$ asserted in Theorem 2, §5.

It follows from known theorems on isotropic functions that each form $\mathfrak{m}_{j_1 \ldots j_k}(\boldsymbol{A}_{j_1}, \ldots, \boldsymbol{A}_{j_k})$, because it is isotropic and multilinear, may be expressed as a sum

$$\mathfrak{m}_{j_1 \ldots j_k}(\boldsymbol{A}_{j_1}, \ldots, \boldsymbol{A}_{j_k}) = \sum_i \boldsymbol{\zeta}_i \tag{7.6}$$

of products of the form

$$\boldsymbol{\zeta}_i = \mu_i \varphi_1 \varphi_2 \ldots \varphi_{\mathfrak{m}_i} (\boldsymbol{A}_{l_1} \boldsymbol{A}_{l_2} \ldots \boldsymbol{A}_{l_{q_i}} + \boldsymbol{A}_{l_{q_i}} \ldots \boldsymbol{A}_{l_2} \boldsymbol{A}_{l_1}) \tag{7.7}$$

where the φ_j's are traces of products of some of the tensors $\boldsymbol{A}_{j_r}$ and are such that each $\boldsymbol{A}_{j_r}$, $r = 1, \ldots, k$, occurs precisely once in each term $\boldsymbol{\zeta}_i$. In the case of an incompressible simple fluid the coefficients μ_i are constants, whereas for simple fluids in general the μ_i are functions of the density. Thus, for simple fluids, a finite number of scalar material functions $p(\varrho)$, $\mu_i(\varrho)$ suffices to determine the stress $\boldsymbol{S}_\alpha$ to within terms of order n in α.

The case $n = 1$ of (7.4) is of particular interest. With use of (7.6), (7.7), we obtain

$$\boldsymbol{S}_\alpha = -p\boldsymbol{I} + \eta \boldsymbol{A}_1^\alpha + \frac{\lambda}{2} (\operatorname{tr} \boldsymbol{A}_1^\alpha) \boldsymbol{I} + o(\alpha). \tag{7.8}$$

Now, the first Rivlin-Ericksen tensor $\boldsymbol{A}_1 = 2\boldsymbol{D}$ differs from the rate of deformation tensor $\boldsymbol{D}$ only by the factor 2*. It follows that (7.8) is, to within terms of order $o(\alpha)$, simply the constitutive equation (1.4) of a Newtonian fluid.

When the fluid is incompressible, the case $n = 2$ of (7.4) takes a remarkably simple form. The observation that $\operatorname{tr} \boldsymbol{A}_1 = 0$ for isochoric motions and use of (7.6), (7.7) yield

$$\boldsymbol{S}_\alpha = -\tilde{p}\boldsymbol{I} + \eta \boldsymbol{A}_1^\alpha + \beta (\boldsymbol{A}_1^\alpha)^2 + \gamma \boldsymbol{A}_2^\alpha + o(\alpha^2), \tag{7.9}$$

where η, β, and γ are material constants and where $\tilde{p}$ is an indeterminate pressure. This pressure $\tilde{p}$ differs, in general, from the mean pressure p defined by the normalization (6.12), because it is obtained from p by absorbing all scalar multiples of $\boldsymbol{I}$ arising from $\mathfrak{m}_{11}$ and $\mathfrak{m}_2$ through use of (7.6) and (7.7).

Motivated by (7.9), we can define an *incompressible second-order fluid* by the constitutive equation

$$\boldsymbol{S} = -\tilde{p}\boldsymbol{I} + \eta \boldsymbol{A}_1 + \beta \boldsymbol{A}_1^2 + \gamma \boldsymbol{A}_2. \tag{7.10}$$

Incompressible Newtonian fluids correspond to the special case $\beta = \gamma = 0$ of (7.10).

In some dynamical situations equation (7.10) leads to a linear partial differential equation for the velocity, just as in the Newtonian case. For example, consider a rectilinear shearing flow which, in Cartesian coordinates x, y, z, has a velocity field with components

$$\{v^i\} = \{0, v(x, t), 0\}. \tag{7.11}$$

If the body forces are conservative, substitution of (7.10) and (7.11) into CAUCHY'S dynamical equations leads to the following third-order partial differential equation for $v(x, t)$:

$$\frac{\partial v}{\partial t} = a \frac{\partial^2 v}{\partial x^2} + b \frac{\partial^3 v}{\partial x^2 \partial t}, \quad a = \frac{\eta}{\varrho}, \quad b = \frac{\gamma}{\varrho}. \tag{7.12}$$

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* *Cf.* [*1*], (9.7).

A non-steady flow of the Couette type, in cylindrical coordinates r, ϑ, z, has a velocity field with contravariant components

$$\{v^i\} = \{0, \omega(r,t), 0\}. \tag{7.13}$$

For such a flow, instead of (7.12), we get

$$r^3 \frac{\partial \omega}{\partial t} = \frac{\partial}{\partial r}\left[r^3\left(a \frac{\partial \omega}{\partial r} + b \frac{\partial^2 \omega}{\partial r\, \partial t}\right)\right]. \tag{7.14}$$

In the Newtonian case, since $b=0$, equations (7.12) and (7.14) reduce to diffusion equations. As for the diffusion equations, many physically interesting solutions of (7.12) and (7.14) may be obtained by separation of variables*. It would be desirable to develop a mathematical theory of third-order partial differential equations of the type (7.12), (7.14).

8. Simple Materials in General

The considerations of the previous two sections may easily be extended to the general simple materials defined in Reference [*1*], Part III.

The form of the constitutive equation of an *isotropic simple material* may be obtained from (6.1) by replacing the scalar density $\varrho(t)$ by the left Cauchy-Green tensor $\boldsymbol{B}(t)$, taken relative to an undistorted reference state (*cf.* [*1*] (22.10)). A consideration analogous to the one which led to (6.7) shows that the constitutive equation of an isotropic simple material may be written in the form

$$\boldsymbol{S}(t) = \mathfrak{h}\big(\boldsymbol{B}(t)\big) + \underset{s=0}{\overset{\infty}{\mathfrak{F}}}\big(\boldsymbol{C}_t(t-s) - \boldsymbol{I}; \boldsymbol{B}(t)\big). \tag{8.1}$$

Here, the functional $\mathfrak{F}$ depends on a tensor parameter $\boldsymbol{B}$, instead of on a scalar parameter ϱ as in (6.7). We assume again that there is an influence function h of order n such that the functional $\mathfrak{F}$ in (8.1) is a memory functional of type (h, n). In place of (7.4), we then obtain the following approximation formula for the stress $\boldsymbol{S}_\alpha$ produced by a slow motion of an isotropic simple material:

$$\boldsymbol{S}_\alpha = \mathfrak{h}(\boldsymbol{B}) + \sum_{(j_1,\dots,j_k)} \mathfrak{m}_{j_1\dots j_k}(\boldsymbol{A}^\alpha_{j_1}, \dots, \boldsymbol{A}^\alpha_{j_k}; \boldsymbol{B}) + o(\alpha^n), \tag{8.2}$$

where $\mathfrak{m}_{j_1\dots j_k}$ is linear in each of its first k tensor variables but not necessarily in the last tensor variable $\boldsymbol{B}$.

The formulae corresponding to (8.1) and (8.2) in the case of *anisotropic simple materials* are obtained from (8.1) and (8.2) simply by replacing all tensors $\boldsymbol{T}$ occurring in these formulae by their conjugates $\boldsymbol{R}^T \boldsymbol{T} \boldsymbol{R}$ with the rotation tensor $\boldsymbol{R} = \boldsymbol{R}(t)$ of the displacement from the reference state (*cf.* [*1*], (22.8)).

In the case of isotropic materials, the function $\mathfrak{h}$ and the functional $\mathfrak{F}$ in (8.1) are isotropic in the sense that they obey the identities

$$\boldsymbol{Q}\, \mathfrak{h}(\boldsymbol{B})\, \boldsymbol{Q}^T = \mathfrak{h}(\boldsymbol{Q}\, \boldsymbol{B}\, \boldsymbol{Q}^T), \tag{8.3}$$

$$\underset{s=0}{\overset{\infty}{\mathfrak{F}}}\big(\boldsymbol{Q}\, \boldsymbol{G}(s)\, \boldsymbol{Q}^T, \boldsymbol{Q}\, \boldsymbol{B}\, \boldsymbol{Q}^T\big) = \boldsymbol{Q} \underset{s=0}{\overset{\infty}{\mathfrak{F}}}\big(\boldsymbol{G}(s), \boldsymbol{B}\big)\, \boldsymbol{Q}^T \tag{8.4}$$

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* In particular, the sinusoidal vibration problems discussed for Newtonian fluids in §§ 345—346 of LAMB's treatise [*5*] are easily solved for second-order fluids. Also, special solutions of (7.14) corresponding to sinusoidal vibrations of a fluid between coaxial cylinders can readily be found.

for all orthogonal tensors $\boldsymbol{Q}$. As in §7, it follows that the $\mathfrak{m}_{j_1 \dots j_k}$ in (8.2) are isotropic functions of all their variables; *i.e.*, they obey the identities

$$\boldsymbol{Q}\,\mathfrak{m}_{j_1 \dots j_k}(\boldsymbol{U}_1, \dots, \boldsymbol{U}_k; \boldsymbol{B})\,\boldsymbol{Q}^T = \mathfrak{m}_{j_1 \dots j_k}(\boldsymbol{Q}\,\boldsymbol{U}_1\,\boldsymbol{Q}^T, \dots, \boldsymbol{Q}\,\boldsymbol{U}_k\,\boldsymbol{Q}^T; \boldsymbol{Q}\,\boldsymbol{B}\,\boldsymbol{Q}^T) \tag{8.5}$$

for all orthogonal $\boldsymbol{Q}$ and all symmetric tensors $\boldsymbol{U}_1, \dots, \boldsymbol{U}_k$, $\boldsymbol{B}$ for which $\mathfrak{m}_{j_1 \dots j_k}$ is defined. The methods developed by SPENCER & RIVLIN ([6], [7] and [8]) may be used to derive explicit representations for the $\mathfrak{m}_{j_1 \dots j_k}$ of a type analogous to but more complicated than (7.6), (7.7). Using such a representation, one can show that, in the case $n = 1$, the approximation formula (8.2) reduces to

$$\boldsymbol{S}_\alpha = \mathfrak{h}(\boldsymbol{B}) + \boldsymbol{A}_1^\alpha\,\mathfrak{k}_1(\boldsymbol{B}) + \mathfrak{k}_1(\boldsymbol{B})\,\boldsymbol{A}_1^\alpha + \operatorname{tr}\left(\boldsymbol{A}_1^\alpha\,\mathfrak{k}_2(\boldsymbol{B})\right)\mathfrak{k}_3(\boldsymbol{B}), + o(\alpha), \tag{8.6}$$

where $\mathfrak{h}$, $\mathfrak{k}_1$, $\mathfrak{k}_2$ and $\mathfrak{k}_3$ are isotropic functions of the one variable $\boldsymbol{B}$ and hence have representations of the form

$$\mathfrak{h}(\boldsymbol{B}) = \beta_0\,\boldsymbol{I} + \beta_1\,\boldsymbol{B} + \beta_2\,\boldsymbol{B}^2, \tag{8.7}$$

in which β_0, β_1 and β_2 are scalar functions of the three principal invariants of $\boldsymbol{B}$.

The first term $\mathfrak{h}(\boldsymbol{B})$ in the expression (8.2) for the stress $\boldsymbol{S}_\alpha$ corresponds to purely elastic response. The sum in (8.2) may be interpreted as representing the internal friction for slow motions.

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References

[1] NOLL, W.: A mathematical theory of the mechanical behavior of continuous media. Arch. Rational Mech. Anal. **2**, 197–266 (1958).

[2] COLEMAN, B. D., & W. NOLL: On certain steady flows of general fluids. Arch. Rational Mech. Anal. **4**, 289–303 (1959).

[3] COLEMAN, B. D., & W. NOLL: Recent results in the continuum theory of viscoelastic fluids. To appear in Ann. New York Acad. Sci.

[4] HILLE, E., & R. S. PHILLIPS: Functional Analysis and Semi-Groups. American Mathematical Society Colloquium Publications, Vol. XXXI. New York 1957.

[5] LAMB, H.: Hydrodynamics, Chap. XI, 6th Edit. Cambridge: Univ. Press 1932.

[6] SPENCER, A. J. M., & R. S. RIVLIN: The theory of matrix polynomials and its application to the mechanics of isotropic continua. Arch. Rational Mech. Anal. **3**, 309–336 (1959).

[7] SPENCER, A. J. M., & R. S. RIVLIN: Finite integrity bases for five or fewer symmetric 3×3 matrices. Arch. Rational Mech. Anal. **3**, 435–446 (1959).

[8] SPENCER, A. J. M., & R. S. RIVLIN: Further results in the theory of matrix polynomials. Arch. Rational Mech. Anal. **4**, 214–230 (1960).

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An Energy Theorem for Viscous Fluid Motions

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1. Introduction

It is intuitively clear that an infinite amount of work is necessary in order to accelerate a body $\mathscr{B}$, initially at rest and surrounded by an incompressible viscous fluid which occupies the (three-dimensional) region exterior to $\mathscr{B}$, into a state of uniform motion with constant velocity, $-\vec{\boldsymbol{w}}_0$. For if $W(t)$ is the work done in accelerating the body from the initial time $t=0$ till the time t, $E(t)$ the kinetic energy of the fluid particles at time t, and $Q(t)$ the rate at which energy is converted into heat in the motion, one has always the relation (see, *e.g.*, [*1*, p. 268]),

$$W(t) = E(t) + \int_0^t Q(\tau)\, d\tau. \tag{1}$$

Letting $\vec{\boldsymbol{u}}(\boldsymbol{x}, t)$ denote the velocity field of the fluid particles at time t and adopting a coordinate frame fixed to $\mathscr{B}$, then the new velocity field, $\vec{\boldsymbol{w}}(\boldsymbol{x}, t) = \vec{\boldsymbol{w}}_0 + \vec{\boldsymbol{u}}(\boldsymbol{x}, t)$ will tend to a steady motion $\vec{\boldsymbol{w}}(\boldsymbol{x})$ which vanishes on the boundary Σ of $\mathscr{B}$ but does not vanish identically. Hence we must have $Q(t) > Q_0 > 0$ for all sufficiently large t, and therefore

$$W(\infty) \geqq \int_{t_0}^{\infty} Q_0\, d\tau = \infty.$$

This reasoning is in two senses incomplete. First, it is not known whether every (or even any) steady motion can be achieved by such a continuous deformation in which the velocity field satisfies at all times the hydrodynamical (Navier-Stokes) equations. The second, and presumably more lasting objection, is that no information is obtained about the relative magnitudes of the two terms on the right side of (1).

We propose in this paper to answer the second question and circumvent the first by studying directly the kinetic energy content of a steady disturbance field defined exterior to $\mathscr{B}$. That is, let $\vec{\boldsymbol{w}}(\boldsymbol{x})$ define a flow in the exterior $\mathscr{E}$ of $\mathscr{B}$, and let $\vec{\boldsymbol{w}}(\boldsymbol{x}) \to \vec{\boldsymbol{w}}_0$ at infinity, $\vec{\boldsymbol{w}}(\boldsymbol{x}) = 0$ on $\mathscr{B}$. We shall show that then the kinetic energy of disturbance, $E = \int_{\mathscr{E}} |\vec{\boldsymbol{w}}(\boldsymbol{x}) - \vec{\boldsymbol{w}}_0|^2\, dV$, is necessarily infinite. This theorem is a relatively easy consequence of results we have proved in [*2*] if suitable assumptions are made on the rate at which $\vec{\boldsymbol{w}}(\boldsymbol{x})$ tends to $\vec{\boldsymbol{w}}_0$. The reader who is content with such a discussion, or who wishes only to see the motivating idea, can refer directly to Theorems 1 and 2 in § 3. Our principal effort will be directed to proving the result under the simplest and most natural assumptions possible (Theorem 6). In this form, the theorem is applicable to the solutions whose existence is demonstrated in [3] and in [4], for which little information about asymptotic behavior is available, and shows that the kinetic energy of disturbance associated with such solutions is necessarily infinite. As additional reward for this effort, we obtain as corollaries of our method several results of independent interest, notably a general theorem on asymptotic behavior of solutions with finite kinetic energy of disturbance (Theorem 4), and a theorem of Liouville type (Theorem 7).

2. General remarks

We shall study time-independent solutions of the Navier-Stokes equations

$$\begin{aligned} \Delta \vec{\boldsymbol{w}} - \vec{\boldsymbol{w}} \cdot \nabla \vec{\boldsymbol{w}} - \nabla p &= 0, \\ \nabla \cdot \vec{\boldsymbol{w}} &= 0. \end{aligned} \tag{2}$$

The notation is the usual one of vector analysis. The velocity field $\vec{\boldsymbol{w}}(\boldsymbol{x})$ and pressure $p(\boldsymbol{x})$, $\boldsymbol{x} = (x_1, x_2, x_3)$, are assumed to possess continuous derivatives of all orders entering in the equations. In (2) we have normalized the density ϱ and viscosity coefficient μ to be unity. This can always be arranged by multiplication of $\vec{\boldsymbol{w}}$ and p by suitable constants, and we assume this done.

Corresponding to a given constant vector $\vec{\boldsymbol{w}}_0$, we introduce the linearized system

$$\begin{aligned} \Delta \vec{\boldsymbol{w}} - \vec{\boldsymbol{w}}_0 \cdot \nabla \vec{\boldsymbol{w}} - \nabla p &= 0, \\ \nabla \cdot \boldsymbol{w} &= 0, \end{aligned} \tag{3}$$

which are the equations satisfied by a perturbation of a solution of (2) about the uniform flow $\vec{\boldsymbol{w}} \equiv \vec{\boldsymbol{w}}_0$. A *fundamental solution tensor* $\boldsymbol{\chi}(\boldsymbol{x}, \boldsymbol{y}) = (\chi_{ij})$, $\boldsymbol{\psi} = (\psi_i)$ corresponding to (3) has been given explicitly by OSEEN [5, p. 34]. For each fixed j (χ_{ij}, ψ_j) defines a solution of (3) in $\boldsymbol{x}$ and a solution of the adjoint system

$$\begin{aligned} \Delta \vec{\boldsymbol{w}} + \vec{\boldsymbol{w}}_0 \cdot \nabla \vec{\boldsymbol{w}} - \nabla p &= 0, \\ \nabla \cdot \vec{\boldsymbol{w}} &= 0 \end{aligned} \tag{4}$$

in $\boldsymbol{y}$. Using this tensor, any solution of (3) in a region $\mathscr{S}$ with boundary Σ can be represented in the form

$$\vec{\boldsymbol{w}}(\boldsymbol{x}) = \vec{\boldsymbol{w}}_0 + \oint_{\Sigma} [\vec{\boldsymbol{u}} \cdot T\boldsymbol{\chi} - \boldsymbol{\chi} \cdot T\vec{\boldsymbol{u}} + (\boldsymbol{\chi} \cdot \vec{\boldsymbol{u}})(\vec{\boldsymbol{w}}_0 \cdot \boldsymbol{n})]\, dS + \int_{\mathscr{S}} \boldsymbol{\chi} \cdot \vec{\boldsymbol{u}} \cdot \nabla \vec{\boldsymbol{u}}\, dV \tag{5}$$

where we have set $\vec{\boldsymbol{u}}(\boldsymbol{x}) = \vec{\boldsymbol{w}}(\boldsymbol{x}) - \vec{\boldsymbol{w}}_0$, and terms of the form $\vec{\boldsymbol{u}} \cdot T\vec{\boldsymbol{v}}$ are to be understood in the sense $\vec{\boldsymbol{u}} \cdot T\vec{\boldsymbol{v}} = u_i (T\vec{\boldsymbol{v}})_{ij} n_j$, with

$$(T\vec{\boldsymbol{v}}) = -p\,\delta_{ij} + \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \qquad \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j, \end{cases}$$

$\boldsymbol{n} = (n_j)$ = unit exterior directed normal, and summation extended over repeated indices. A similar representation is available for the pressure,

$$p = p_0 + \oint_{\Sigma} [\vec{\boldsymbol{u}} \cdot T\boldsymbol{\psi} - \boldsymbol{\psi} \cdot T\vec{\boldsymbol{u}} + (\boldsymbol{\psi} \cdot \boldsymbol{u})(\vec{\boldsymbol{w}}_0 \cdot \boldsymbol{n})]\, dS + \int_{\mathscr{S}} \boldsymbol{\psi} \cdot \vec{\boldsymbol{u}} \cdot \nabla \vec{\boldsymbol{u}}\, dV \tag{6}$$

where we have introduced a "pressure" $\psi^* = \vec{\boldsymbol{w}}_0 \cdot \nabla \left(\frac{1}{r_{\boldsymbol{xy}}}\right)$ corresponding to the vector $\boldsymbol{\psi}(\boldsymbol{x}, t)$.

In the case that the region is a sphere V and $\vec{\boldsymbol{w}}_0 = 0$, we shall need the *Green's tensor* $\boldsymbol{G}(\boldsymbol{x}, \boldsymbol{y})$, $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y})$ associated with (3). This tensor has also been given explicitly by OSEEN [5, p. 105]. Using it, we find

$$\begin{aligned} \vec{\boldsymbol{w}}(\boldsymbol{x}) &= \oint_{\Sigma} \vec{\boldsymbol{w}} \cdot T\boldsymbol{G}\, dS + \int_{V} \boldsymbol{G} \cdot \vec{\boldsymbol{w}} \cdot \nabla \vec{\boldsymbol{w}}\, dV, \\ p(\boldsymbol{x}) &= \oint_{\Sigma} \vec{\boldsymbol{w}} \cdot T\boldsymbol{P}\, dS + \int_{V} \boldsymbol{P} \cdot \vec{\boldsymbol{w}} \cdot \nabla \vec{\boldsymbol{w}}\, dV. \end{aligned} \tag{7}$$

The formulas of OSEEN yield the important estimates

$$\begin{aligned} |\boldsymbol{\chi}(\boldsymbol{x}, \boldsymbol{y})| = \max_{i,j} |\chi_{ij}| < C\, r_{\boldsymbol{x}\boldsymbol{y}}^{-1}, &\quad |\nabla \boldsymbol{\chi}| < C\, r_{\boldsymbol{x}\boldsymbol{y}}^{-\frac{3}{2}}, \\ |\boldsymbol{\psi}(\boldsymbol{x}, \boldsymbol{y})| < C\, r_{\boldsymbol{x}\boldsymbol{y}}^{-2}&, \\ |\nabla_{\boldsymbol{y}} \boldsymbol{\psi}(\boldsymbol{x}, \boldsymbol{y})| < C\, r_{\boldsymbol{x}\boldsymbol{y}}^{-3}, &\quad |T\boldsymbol{\psi}| < C\, r_{\boldsymbol{x}\boldsymbol{y}}^{-2}, \end{aligned} \tag{8}$$

and for any fixed sphere,

$$|\boldsymbol{G}(\boldsymbol{x}, \boldsymbol{y})| < C\, r_{\boldsymbol{x}\boldsymbol{y}}^{-1}, \quad |\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y})| < C\, r_{\boldsymbol{x}\boldsymbol{y}}^{-2}, \quad |\nabla \boldsymbol{G}(\boldsymbol{x}, \boldsymbol{y})| < C\, r_{\boldsymbol{x}\boldsymbol{y}}^{-2} \tag{9}$$

uniformly in x and y, for some fixed constant C.

Throughout this paper, the symbol C will be used to denote a constant, the value of which may change in the context. Thus, from the inequality $\alpha \leqq C\sqrt{1 + \beta^2}$ we may conclude $\alpha \leqq C|\beta|$ as β varies over some range. Analogously, we denote by ε such a quantity which is known also to be small in magnitude.

3. Preliminary Estimates

Lemma 1. *Let* $\vec{\boldsymbol{w}}(\boldsymbol{x})$, $p(\boldsymbol{x})$ *define a solution of* (2) *in the exterior of a smooth closed surface* Σ, *and suppose that for some constant vector* $\vec{\boldsymbol{w}}_0$, $\vec{\boldsymbol{u}}(\boldsymbol{x}) = \vec{\boldsymbol{w}}(\boldsymbol{x}) - \vec{\boldsymbol{w}}_0$ *satisfies the inequalities,*

$$\int_{\mathscr{E}} |\vec{\boldsymbol{u}}|^2\, dV < \infty, \quad |\vec{\boldsymbol{u}}(\boldsymbol{x})| < M < \infty,$$

uniformly for all $\boldsymbol{x}$ *in* $\mathscr{E}$. *Then* $\vec{\boldsymbol{u}}(\boldsymbol{x}) \to 0$ *as* $\boldsymbol{x} \to \infty$.

Proof. Let ε be an arbitrary positive quantity. Choose R_0 so large that in the region $\mathscr{E}_{R_0}$ exterior to a sphere about the origin of radius R_0, $\int_{\mathscr{E}_{R_0}} |\vec{\boldsymbol{u}}|^2\, dV < \varepsilon$. Let $\boldsymbol{x}$ denote an arbitrary point such that $|\boldsymbol{x}| > R_0 + 4$, and let V_a be a sphere centered at $\boldsymbol{x}$ and of radius a to be determined in the interval $1 \leqq a \leqq 2$.

We have

$$\vec{\boldsymbol{u}}(\boldsymbol{x}) = \int_{V_a} \boldsymbol{G} \cdot \vec{\boldsymbol{u}} \cdot \nabla \vec{\boldsymbol{u}}\, dV + \int_{V_a} \boldsymbol{G} \cdot \vec{\boldsymbol{w}}_0 \cdot \nabla \vec{\boldsymbol{u}}\, dV + \oint_{\Sigma_a} \vec{\boldsymbol{u}} \cdot T\boldsymbol{G}\, dS,$$

Σ_a being the bounding surface of V_a. An integration by parts, using the fact that $\boldsymbol{G}(\boldsymbol{x}, \boldsymbol{y})$ vanishes on Σ_a, yields

$$\vec{\boldsymbol{u}}(\boldsymbol{x}) = \int_{V_a} \vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{u}} \cdot \nabla \boldsymbol{G}\, dV - \int_{V_a} \vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{w}}_0 \cdot \nabla \boldsymbol{G}\, dV + \oint_{\Sigma_a} \vec{\boldsymbol{u}} \cdot T\boldsymbol{G}\, dS.$$

From the estimates (9) and SCHWARZ' Inequality we find

$$|\vec{\boldsymbol{u}}(\boldsymbol{x})| \leqq \int_{V_a} \frac{|\vec{\boldsymbol{u}}|^2}{r_{\boldsymbol{x}\boldsymbol{y}}^2}\, dV_{\boldsymbol{y}} + |\vec{\boldsymbol{w}}_0| \int_{V_a} \frac{|\vec{\boldsymbol{u}}|}{r_{\boldsymbol{x}\boldsymbol{y}}^2}\, dV + C\left[\oint_{\Sigma_a} |\vec{\boldsymbol{u}}|^2\, dS\right]^{\frac{1}{2}}.$$

By the choice of R_0, $\int\limits_1^2 dr \oint\limits_{\Sigma_r} |\vec{u}|^2\, dS < \varepsilon$, hence we may choose an a in the given range such that $\oint\limits_{\Sigma_a} |\vec{u}|^2\, dS < \varepsilon$. Using the second hypothesis, we obtain

$$|\vec{u}(\boldsymbol{x})| < C \int\limits_{V_a} \frac{|\vec{u}|}{r_{\boldsymbol{x}\boldsymbol{y}}^2}\, dV_{\boldsymbol{y}} + \varepsilon. \tag{10}$$

Similarly, for some b in the range $3 \leq b \leq 4$ we obtain

$$|\vec{\ }(\boldsymbol{x})| < C \int\limits_{V_b} \frac{|\vec{u}|}{r_{\boldsymbol{x}\boldsymbol{y}}^2}\, dV_{\boldsymbol{y}} + \varepsilon.$$

This latter inequality is valid uniformly for all x interior to V_a. Multiplying by $r_{\boldsymbol{x}\boldsymbol{z}}^{-2}$ and integrating over V_a gives

$$\int\limits_{V_a} \frac{|\vec{u}(\boldsymbol{x})|}{r_{\boldsymbol{x}\boldsymbol{z}}^2}\, dV_{\boldsymbol{x}} < C \int\limits_{V_b} \frac{|\vec{u}(\boldsymbol{y})|}{r_{\boldsymbol{y}\boldsymbol{z}}}\, dV_{\boldsymbol{y}} + \varepsilon < C \Bigl(\int\limits_{V_b} |\vec{u}(\boldsymbol{y})|^2\, dV_{\boldsymbol{y}} \Bigr)^{\frac{1}{2}} + \varepsilon < \varepsilon,$$

by SCHWARZ' Inequality. Inserting this result in (10) completes the proof of the lemma.

Lemma 2. *Let $\vec{\boldsymbol{w}}(\boldsymbol{x})$, $p(\boldsymbol{x})$ define a solution of* (2) *in a neighborhood $\mathcal{E}$ of infinity, and suppose $\vec{\boldsymbol{w}}(\boldsymbol{x}) \to \vec{\boldsymbol{w}}_0$ as $\boldsymbol{x} \to \infty$. Then $\nabla \vec{\boldsymbol{w}} \to 0$ and $p = o(\log r)$ as $\boldsymbol{x} \to \infty$.*

These results are proved in Theorems 1 and 2 of [*2*].

Lemma 3. *Under the hypotheses of Lemma 2, suppose in addition that $\int\limits_{\mathcal{E}} |\vec{u}|^2\, dV < \infty$, $\vec{u} = \vec{\boldsymbol{w}} - \vec{\boldsymbol{w}}_0$. Then there is a constant p_0 such that $p(\boldsymbol{x}) \to p_0$ as $\boldsymbol{x} \to \infty$. If Σ is any smooth closed surface which together with its exterior lies in $\mathcal{E}$, then*

$$p(\boldsymbol{x}) = p_0 + \oint\limits_{\Sigma} [\vec{u} \cdot T\psi - \psi \cdot T\vec{u} + (\psi \cdot \vec{u})(\vec{w}_0 \cdot n)]\, dS + \int\limits_{\mathcal{E}} \psi \cdot \vec{u} \cdot \nabla \vec{u}\, dV, \tag{11}$$

$\mathcal{E}$ denoting now the exterior of Σ.

Proof. By SCHWARZ' Inequality, the estimates (8), and Lemma 2, we see that the volume integral in (11) is absolutely convergent. Further, this term tends to zero as $\boldsymbol{x} \to \infty$. To see this, let $\mathcal{S}_R$ be the annular region bounded by Σ and by a sphere Σ_R about the origin of (large) radius R, and let $\mathcal{E}_R$ be the exterior of Σ_R. Let V be a unit sphere centered at $\boldsymbol{x}$. Then for $|\boldsymbol{x}| > R + 1$, $\int\limits_{\mathcal{E}} = \int\limits_{\mathcal{S}_R} + \int\limits_{V} + \int\limits_{\mathcal{E}_R - V}$. For given $\varepsilon > 0$, we may choose R so large that $\int\limits_{\mathcal{E}_R} |\vec{u}|^2\, dV < \varepsilon$. Then, because of the estimates (8) and Lemma 2, we may choose $|\boldsymbol{x}|$ so large that $\Bigl| \int\limits_{\mathcal{S}_R} \psi \cdot \vec{u} \cdot \nabla \vec{u}\, dV \Bigr| < \varepsilon$ and also $\Bigl| \int\limits_{V} \psi \cdot \vec{u}\, \nabla \vec{u}\, dV \Bigr| < \varepsilon$. For this choice of R and $\boldsymbol{x}$, we have

$$\Bigl| \int\limits_{\mathcal{E}_R - V} \psi \cdot \vec{u} \cdot \nabla \vec{u}\, dV \Bigr|^2 \leq C \int\limits_{\mathcal{E}_R} |\vec{u}|^2\, dV \int\limits_1^\infty \frac{1}{r^2}\, dr < \varepsilon,$$

and hence $\Bigl| \int\limits_{\mathcal{E}} \psi \cdot \boldsymbol{u} \cdot \nabla \boldsymbol{u}\, dV \Bigr| < \varepsilon$, the desired result.

In the (finite) region $\mathscr{S}_R$, we have by (6)

$$p(\boldsymbol{x}) = \oint_{\Sigma} [\vec{u} \cdot T\boldsymbol{\psi} - \boldsymbol{\psi} \cdot T\vec{u} + (\boldsymbol{\psi} \cdot \vec{u})\,(\vec{w}_0 \cdot \boldsymbol{n})]\, dS + \oint_{\Sigma_R} [\ \ ]\, dS + \int_{\mathscr{S}_R} \boldsymbol{\psi} \cdot \vec{u} \cdot \nabla \vec{u}\, dV.$$

Fixing $\boldsymbol{x}$ and letting $R \to \infty$, we see that of the two terms which depend on R, the latter tends to a limit, hence so must the former. Set

$$F(\boldsymbol{x}) = \lim_{R\to\infty} \oint_{\Sigma_R} [\vec{u} \cdot T\boldsymbol{\psi} - \boldsymbol{\psi} \cdot T\vec{u} + (\boldsymbol{\psi} \cdot \vec{u})\,(\vec{u} \cdot \boldsymbol{n})]\, dS.$$

In this integral, all terms clearly tend to zero with the possible exception of the middle one, which involves the pressure. Let

$$H(\boldsymbol{x}) = \lim_{R\to\infty} \int_R \boldsymbol{\psi} \cdot T\vec{u}\, dS,$$

and denote by $\delta^h H(\boldsymbol{x})$ a difference quotient corresponding to a displacement h in an arbitrary direction. A simple application of the mean value theorem shows that

$$\delta^h H(\boldsymbol{x}) = \lim_{R\to\infty} \oint \delta^h \boldsymbol{\psi} \cdot T\vec{u}\, dS \equiv 0$$

in $\boldsymbol{x}$, by (8) and Lemma 2. Since h is arbitrary, we conclude that $H(\boldsymbol{x}) \equiv F(\boldsymbol{x}) \equiv$ const $= p_0$, and Lemma 3 follows directly.

Theorem 1. *Let $\vec{w}(\boldsymbol{x})$ be a solution of* (2) *exterior to a smooth closed surface Σ, and suppose that for some constant vector $\vec{w}_0 \neq 0$ and some $\varepsilon > 0$, $|\vec{w}(\boldsymbol{x}) - \vec{w}_0| < C\, r^{-\frac{1}{2}-\varepsilon}$ as $\boldsymbol{x} \to \infty$.*

Let[*]

$$\boldsymbol{a} = \oint_{\Sigma} \{T\vec{w} - \vec{w}(\vec{w} \cdot \boldsymbol{n}) - \vec{w}_0(\vec{w} \cdot \boldsymbol{n})\}\, dS.$$

Then if $\vec{u}(\boldsymbol{x}) = \vec{w}(\boldsymbol{x}) - \vec{w}_0$, we shall have $\int_{\mathscr{E}} |\vec{u}(\boldsymbol{x})|^2\, dV < \infty$ if and only if $\boldsymbol{a} = 0$.

Proof. By Theorem 8 of [2],

$$\vec{u}(\boldsymbol{x}) = \vec{u}_{\Sigma}(\boldsymbol{x}) + \vec{u}_{\mathscr{E}}(\boldsymbol{x}),$$

with

$$\vec{u}_{\Sigma}(\boldsymbol{x}) = \oint_{\Sigma} \{\vec{u} \cdot T\boldsymbol{\chi} - \boldsymbol{\chi} \cdot T\vec{u} + (\boldsymbol{\chi} \cdot \vec{w})\,(\vec{w} \cdot \boldsymbol{n}) + (\boldsymbol{\chi} \cdot \vec{w})\,(\vec{w} \cdot \boldsymbol{n})\}\, dS,$$

and $\vec{u}_{\mathscr{E}}(\boldsymbol{x})$, as is easily verified from the estimate (27) of [2], has finite square integral over $\mathscr{E}$. A simple application of the mean value theorem, using the estimates (8), shows that for large values of $|\boldsymbol{x}|$,

$$\vec{u}_{\Sigma}(\boldsymbol{x}) = -\boldsymbol{a} \cdot \boldsymbol{\chi}(\boldsymbol{x}, 0) + \vec{u}_1(\boldsymbol{x}),$$

where $\vec{u}_1(\boldsymbol{x})$ is square integrable over $\mathscr{E}$. (The point $\boldsymbol{y} = 0$ is assumed interior to Σ.) Thus, it suffices to show that $\boldsymbol{a} \cdot \boldsymbol{\chi}(\boldsymbol{x}, 0)$ has infinite square integral whenever $\boldsymbol{a} \neq 0$. Set $\boldsymbol{a} = (a_1, a_2, a_3)$. Then $|\boldsymbol{a} \cdot \boldsymbol{\chi}(\boldsymbol{x}, 0)|^2 = \chi_{ij}\,\chi_{ik}\,a_j\,a_k$, a quadratic form in the components of $\boldsymbol{a}$. We integrate this form termwise over a sphere Σ_R about the origin, and observe that by the symmetry properties of χ_{ij} (see, *e.g.*, the defining relations [5, p. 34] of OSEEN) all terms for which $j \neq k$ integrate to zero. Thus, we need only study the sum of squares $\chi_{ij}^2 a_j^2$ for each of the indices

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[*] The vector $\boldsymbol{a}$ has a physical significance as the sum of the force exerted on Σ by the fluid, the flux of momentum across Σ, and a momentum flux arising from the motion at infinity.

$i=1, 2, 3$ (no summation on i). Tedious but formal estimation yields

$$\oint_{\Sigma_R} \chi_{11}^2 \, dS > C R^{-1}, \quad \oint_{\Sigma_R} \chi_{22}^2 \, dS > C R^{-1}, \quad \oint_{\Sigma_R} \chi_{33}^2 \, dS > C R^{-1},$$

for some positive C, and from this the theorem follows.

Theorem 2. *Let $\vec{w}(\boldsymbol{x})$ be a solution of (2) exterior to a smooth closed surface Σ, and suppose that $|\vec{w}(\boldsymbol{x}) - \vec{w}_0| < C r^{-\frac{1}{2}-\varepsilon}$ as $\boldsymbol{x} \to \infty$, $\vec{w}_0 \neq 0$. If $\vec{w}(\boldsymbol{x}) = 0$ for $\boldsymbol{x}$ on Σ, then necessarily $\oint_{\mathscr{E}} |\vec{u}(\boldsymbol{x})|^2 \, dV = \infty$.*

Proof. If $\mathfrak{F}$ is the force on Σ, then by Theorem 13 of [2],

$$\frac{1}{2} \int_{\mathscr{E}} \left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i} \right)^2 dV = \vec{w}_0 \cdot \mathfrak{F}.$$

It follows that $\mathfrak{F} \neq 0$, and the result is obtained directly from Theorem 1.

Lemma 4. *Let $\boldsymbol{\gamma}(\boldsymbol{x})$ be a vector valued function defined and piecewise continuously differentiable* throughout space. Then there exists a constant vector $\boldsymbol{\gamma}_0$ such that, for any choice of the origin of coordinates,*

$$\int_{\mathscr{E}} \frac{|\boldsymbol{\gamma} - \boldsymbol{\gamma}_0|^2}{r^2} \, dV \leq 4 \int_{\mathscr{E}} |\nabla \boldsymbol{\gamma}|^2 \, dV,$$

the integration being taken over the entire space $\mathscr{E}$.

This lemma is proved as Lemma 3.4 in [6].

4. The Main Theorems

Theorem 3. *Let $\vec{w}(\boldsymbol{x})$ be a solution of (2) in the exterior $\mathscr{E}$ of a smooth closed surface Σ, and suppose that for some constant vector $\vec{w}_0$, $\vec{u}(\boldsymbol{x}) = \vec{w}(\boldsymbol{x}) - \vec{w}_0$ satisfies the inequalities*

$$\int_{\mathscr{E}} |\vec{u}|^2 \, dV < \infty, \quad |\vec{u}(\boldsymbol{x})| < M < \infty,$$

uniformly for all $\boldsymbol{x}$ in $\mathscr{E}$. Then $\vec{w}(\boldsymbol{x})$ has finite Dirichlet Integral over $\mathscr{E}$, $\int_{\mathscr{E}} |\nabla \vec{w}|^2 \, dV < \infty$. Denoting the value of this integral by A and the contribution from the annular region $\mathscr{S}_R$ bounded by Σ and by a sphere Σ_R of (large) radius R by $D(R)$, we have

$$A = \oint_{\Sigma} \left\{ \vec{u} \cdot \frac{\partial \vec{u}}{\partial n} - |\vec{u}|^2 (\vec{u} \cdot \boldsymbol{n}) - |\vec{u}|^2 (\vec{w}_0 \cdot \boldsymbol{n}) - p (\vec{u} \cdot \boldsymbol{n}) \right\} dS$$

and

$$|A - D(R)| < C R^{-2}.$$

Proof. Lemmas 1, 2, and 3 show that $\vec{w}(\boldsymbol{x}) \to \vec{w}_0$, $\nabla \vec{w}(\boldsymbol{x}) \to 0$, and $p(\boldsymbol{x}) \to p_0$ as $\boldsymbol{x} \to \infty$. We shall assume, without loss of generality, that $p_0 = 0$. Formal integration by parts yields the relation

$$D(R) - A = \oint_{\Sigma_R} \left\{ \vec{u} \cdot \frac{\partial \vec{u}}{\partial n} - |\vec{u}|^2 (\vec{u} \cdot \boldsymbol{n}) - |\vec{u}|^2 (\vec{w}_0 \cdot \boldsymbol{n}) - p (\vec{u} \cdot \boldsymbol{n}) \right\} dS. \tag{12}$$

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* That is, continuous throughout space and possessing bounded derivatives which are continuous except perhaps on a finite number of smooth surfaces.

Set $H(r)=\oint_{\Sigma_R}|\vec{u}|^2 d\,S$, $K(r)=\int_r^{r+1} H(r)\,d\,r$, and let us denote an interval of the form $[r, 2r]$ by I_r. There must exist an r_0 such that for every integer $r>r_0$ at least half of the values $K(r_j)$, $r_j=r$, $r+1$, $\ldots$, $2r-1$ which occur in any I_r satisfy the inequality $K(r_j)<\frac{1}{r_j}$. For otherwise, there would be an infinity of disjoint intervals $\{I_r\}$, on each of which at least half the corresponding $K(r_j)$ would satisfy $K(r_j)>\frac{1}{r_j}>\frac{1}{2r}$. We would conclude $\int_{I_r} H(r)\,d\,r>\frac{r}{4r}=\frac{1}{4}$ for each such I_r, and hence $\int_{\mathscr{E}}|\vec{u}|^2 d\,V=\infty$, a contradiction.

Denote by $\{E_j\}$ the set of such disjoint unit intervals $E_j=[r_j, r_{j+1}]$ for which $K(r_j)<\frac{1}{r_j}$. On each such interval there must be a set S_j of measure $\mu(S_j)\geqq\frac{1}{2}$, on which $H(r)<\frac{2}{r_j}$. It follows that there is a fixed $\delta_0>0$ such that on each E_j there are points r_1, r_2, and a set S_j^0 of measure $\geqq\delta_0$ contained between these points and of distance $\geqq\delta_0$ from each of them such that on S_j^0, $H<\frac{2}{r_j}$.

Consider now $\oint_{\Sigma_r} p\,(\vec{u}\cdot\boldsymbol{n})\,d\,S$ for r in some S_j^0. Since $p\to 0$, we have by SCHWARZ' Inequality,

$$\left(\oint_{\Sigma_r} p\,(\vec{u}\cdot\boldsymbol{n})\,d\,S\right)^2\leqq\varepsilon\oint_{\Sigma_r}|p|\,d\,S\oint_{\Sigma_r}|\vec{u}|^2 d\,S \tag{13}$$
$$\leqq\frac{\varepsilon}{r_j}\oint_{\Sigma_r}|p|\,d\,S.$$

We proceed to estimate the integral on the right, using the representation (11) for $p(\boldsymbol{x})$:

$$p(\boldsymbol{x})=\oint_{\Sigma}[\vec{u}\cdot T\boldsymbol{\psi}-\boldsymbol{\psi}\cdot T\vec{u}-(\boldsymbol{\psi}\cdot\vec{u})\,(\vec{\boldsymbol{w}}_0\cdot\boldsymbol{n})]\,d\,S+\int_{\mathscr{E}}\boldsymbol{\psi}\cdot\vec{u}\cdot\nabla\vec{u}\,d\,V$$
$$=\int_{\mathscr{E}}\boldsymbol{\psi}\cdot\vec{u}\cdot\nabla\vec{u}\,d\,V+O\left(\frac{1}{r^2}\right).$$

In the interval E_j, we consider the subinterval $[r_1, r_2]$, and deform the volume integral by an integration by parts:

$$p(\boldsymbol{x})=\int_{\mathscr{A}_r}\boldsymbol{\psi}\cdot\vec{u}\cdot\nabla\vec{u}\,d\,V+\oint_{\Sigma_{r_1}+\Sigma_{r_2}}(\boldsymbol{\psi}\cdot\vec{u})\,(\vec{u}\cdot\boldsymbol{n})\,d\,S-\int_{\mathscr{E}-\mathscr{A}_r}\vec{u}\cdot\vec{u}\cdot\nabla\boldsymbol{\psi}\,d\,V+O\left(\frac{1}{r^2}\right), \tag{14}$$

where $\mathscr{A}_r$ is the annular region bounded by Σ_{r_1} and Σ_{r_2}. We evaluate the integral of $|p|$ over Σ_r by integrating (14) under the sign. We must therefore study

$$\oint_{\Sigma_r}|\boldsymbol{\psi}(\boldsymbol{x},\boldsymbol{y})|\,d\,S_{\boldsymbol{x}}\quad\text{and}\quad\oint_{\Sigma_r}|\nabla\boldsymbol{\psi}|\,d\,S_{\boldsymbol{x}}.$$

We distinguish cases (a), (b), (c), according as $\boldsymbol{y}$ lies in $\mathscr{A}_r$, on Σ_{r_1} or Σ_{r_2}, or in $\mathscr{E}-\mathscr{A}_r$.

Case (a). Let ϱ_0 be the distance from $\boldsymbol{y}$ to Σ_r, ϱ the distance from $\boldsymbol{y}$ to $\boldsymbol{x}$. Using (8), we obtain

$$|\boldsymbol{\psi}(\boldsymbol{x},\boldsymbol{y})|<\frac{C}{\varrho^2}=\frac{C}{r^2+(r+\varrho_0)-2r(r+\varrho_0)\cos\varphi},$$

φ being the polar angle measured from the origin, with the line joining the origin to y as axis. Thus,

$$\oint_{\Sigma_r} |\psi(\boldsymbol{x}, \boldsymbol{y})|\, d S_{\boldsymbol{x}} < C \log \frac{r}{\varrho_0}$$

when r is large compared to ϱ_0. The contribution to $\oint_{\Sigma_r} |p|\, d S$ arising from $\mathscr{A}_r$ is thus

$$\left| \int_{\mathscr{A}_r} \vec{\boldsymbol{u}} \cdot \nabla \vec{\boldsymbol{u}}\, d V \oint_{\Sigma_r} \psi\, d S \right| \leqq \varepsilon \sqrt{\int_{\mathscr{A}_r} \log^2 \frac{r}{\varrho_0}\, d V} \sqrt{\int_{\mathscr{A}_r} |\vec{\boldsymbol{u}}|^2 d V}$$

$$\leqq \varepsilon\, r \log r \cdot \sqrt{\frac{1}{r}} = \varepsilon \sqrt{r} \log r$$

by the choice of the interval E_j.

Case (b). We have

$$\left| \oint_{\Sigma_r} d S \oint_{\Sigma_{r_1}} (\psi \cdot \vec{\boldsymbol{u}})\, (\vec{\boldsymbol{u}} \cdot \boldsymbol{n})\, d S \right| \leqq C \log \frac{r}{\delta_0} \oint_{\Sigma_{r_1}} |\vec{\boldsymbol{u}}|^2 d S < \frac{C}{r} \log \frac{r}{\delta_0}$$

with a similar estimate for Σ_{r_2}.

Case (c). Since $|\nabla \psi| < C/r^3_{\boldsymbol{x}\boldsymbol{y}}$, we find for $\boldsymbol{y}$ in $\mathscr{E} - \mathscr{A}_r$, $\oint_{\Sigma_r} |\nabla \psi|\, d S < C/\delta_0$, hence

$$\left| \oint_{\Sigma_r} d S \int_{\mathscr{E} - \mathscr{A}_r} \vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{u}} \cdot \nabla \psi\, d V \right| < \frac{C}{\delta_0} \int_{\mathscr{E}} |\vec{\boldsymbol{u}}|^2 d V < \frac{C}{\delta_0}.$$

Collecting these results and inserting them in (13) yields

$$\left| \oint_{\Sigma_r} p\, (\vec{\boldsymbol{u}} \cdot \boldsymbol{n})\, d S \right|^2 \leqq \frac{\varepsilon}{\sqrt{r}} \log r \leqq \varepsilon,$$

for r in some S^0_j. Hence we obtain from (12) that for all such r,

$$|D(r) - A| \leqq \sqrt{\oint_{\Sigma_r} |\vec{\boldsymbol{u}}|^2 d S} \sqrt{\oint_{\Sigma_r} |\nabla \vec{\boldsymbol{u}}|^2 d S} + C \oint_{\Sigma_r} |\vec{\boldsymbol{u}}|^2 d S + \varepsilon,$$

and therefore

$$|D(r) - A|^2 \leqq \frac{C}{r} \frac{dD(r)}{dr} + \varepsilon \tag{15}$$

whenever r is in an S^0_j.

Suppose now that for some R_1, $D(R_1) > A$. Then since $D(r)$ is non-decreasing, we obtain from (15)

$$\frac{1}{|D(r) - A|^2} \frac{dD(r)}{dr} \geqq \begin{cases} r(1 - \varepsilon) & \text{if} \quad r \in S^0_j \\ 0 & \text{elsewhere}. \end{cases} \tag{16}$$

Now in every interval $I_r = [r, 2r]$, at least half the unit subintervals, *i.e.*, the intervals E_j, contain a subset S^0_j of measure at least δ_0, δ_0 independent of r. Integration of (16) from R_1 to R thus yields

$$\frac{1}{D(R_1) - A} - \frac{1}{D(R) - A} \geqq C\, \delta_0 (R^2 - R_1^2).$$

This is utter nonsense, as we see by letting $R \to \infty$, so it must be that $D(r) < A$, all r, and therefore

$$\frac{1}{A - D(r)} \frac{dD(r)}{dr} \geqq \begin{cases} r(1-\varepsilon) & \text{if} \quad r \in S_j^0 \\ 0 & \text{elsewhere.} \end{cases}$$

Integration now yields

$$\frac{1}{A - D(R)} - \frac{1}{A - D(R_1)} \geqq C\,\delta_0 (R^2 - R_1^2)$$

from which

$$A - D(R) \leqq C\,R^{-2}$$

for all sufficiently large R. This completes the proof of Theorem 3.

Theorem 4. *Under the hypotheses of Theorem 3, if* $\vec{\boldsymbol{w}}_0 \neq 0$, *then* $|\vec{\boldsymbol{w}}(\boldsymbol{x}) - \vec{\boldsymbol{w}}_0| < C r^{-\frac{3}{2}+\varepsilon}$ *as* $\boldsymbol{x} \to \infty$, *for any* $\varepsilon > 0$.

Proof. For $\boldsymbol{x}$ in $\mathscr{E}$, the representation (5) gives

$$\vec{\boldsymbol{u}}(\boldsymbol{x}) = -\oint_{\Sigma} [\vec{\boldsymbol{u}} \cdot T\boldsymbol{\chi} - \boldsymbol{\chi} \cdot T\vec{\boldsymbol{u}} + (\boldsymbol{\chi} \cdot \vec{\boldsymbol{u}})(\vec{\boldsymbol{w}}_0 \cdot \boldsymbol{n})]\, dS - \int_{\mathscr{E}} \boldsymbol{\chi} \cdot \vec{\boldsymbol{u}}\, \nabla \vec{\boldsymbol{u}}\, dV.$$

(A discussion analogous to the one at the end of the proof of Lemma 3 shows that the other boundary integral vanishes in the limit.) The estimates (8) show that the surface integral over Σ has the behavior $O(r^{-1})$ at infinity. To study the volume integral, we let $|\boldsymbol{x}| = 2R$ and introduce the region $\mathscr{S}_R$, bounded by Σ and by a sphere about the origin of radius R. For all $\boldsymbol{y}$ in $\mathscr{S}_R$, $|\boldsymbol{\chi}(\boldsymbol{x}, \boldsymbol{y})| < \frac{C}{R}$ by (8). Hence, by hypothesis and by Theorem 3,

$$\begin{aligned} \left| \int_{\mathscr{S}_R} \boldsymbol{\chi} \cdot \vec{\boldsymbol{u}} \cdot \nabla \vec{\boldsymbol{u}}\, dV \right| &< C R^{-1} \sqrt{\int_{\mathscr{S}_R} |\vec{\boldsymbol{u}}|^2\, dV} \sqrt{\int_{\mathscr{S}_R} |\nabla \boldsymbol{u}|^2\, dV} \\ &< C R^{\;1}. \end{aligned}$$

Finally, setting $\mathscr{E}_R = \mathscr{E} - \mathscr{S}_R$,

$$\begin{aligned} \left| \int_{\mathscr{E}_R} \boldsymbol{\chi} \cdot \vec{\boldsymbol{u}} \cdot \nabla \vec{\boldsymbol{u}}\, dV \right| &< C \sqrt{\int_{\mathscr{E}} \frac{|\vec{\boldsymbol{u}}|^2}{r_{\boldsymbol{x}\boldsymbol{y}}^2}\, dV} \sqrt{\int_{\mathscr{E}_R} |\nabla \boldsymbol{u}|^2\, dV} \\ &< C R^{-1} \end{aligned}$$

by Lemma 4 and Theorem 3. Thus $|\vec{\boldsymbol{u}}(\boldsymbol{x})| < C r^{-1}$. The method of proof of Theorem 1 now shows that the vector $\boldsymbol{a}$ must vanish, and the stated result then follows from Theorem 8 of [2]*.

From Theorems 1, 2, 3, and 4 we obtain the following results:

Theorem 5. *Under the hypotheses of Theorem 3, if there is no flux of fluid or momentum flux across* Σ, *then the force exerted on* Σ *by the fluid is zero.*

If $\vec{\boldsymbol{w}}_0 \neq 0$, Theorem 5 follows from Theorems 1 and 4. But if $\vec{\boldsymbol{w}}_0 = 0$, we have the following lemma:

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* One sees easily that if in addition there is no net flux of fluid across Σ, then $\vec{\boldsymbol{u}}(\boldsymbol{x})$ satisfies the stronger estimate (27) of [2]. For the application of Theorem 4 to the later results of this paper, the estimate $|\vec{\boldsymbol{u}}| < C r^{-1}$ already suffices.

Lemma 5. *Under the hypotheses of Theorem 3 if* $\vec{w}_0 = 0$, *then*

$$\vec{w}(x) = \oint_{\Sigma} \{\chi \cdot T\vec{w} - \vec{w} \cdot T\chi - (\chi \cdot \vec{w})(\vec{w} \cdot \vec{n})\}\, dS + O\left(\frac{1}{r^2}\right). \tag{17}$$

Proof. As in the demonstration of Theorem 4, we have

$$\vec{w}(x) = -\oint_{\Sigma} [\vec{w} \cdot T\chi - \chi \cdot Tw]\, dS - \int_{\mathscr{E}} \chi \cdot \vec{w} \cdot \nabla w\, dV.$$

An integration by parts changes this expressions to

$$\vec{w}(x) = -\oint_{\Sigma} [\vec{w} \cdot T\chi - \chi \cdot T\vec{w} + (\chi \cdot \vec{w})(\vec{w} \cdot n)]\, dS + \int_{\mathscr{E}} \vec{w} \cdot \vec{w} \cdot \nabla\chi\, dV.$$

Again we evaluate the volume integral separately in the two regions $\mathscr{S}_R$ and $\mathscr{E}_R$. Since★ for $\vec{w}_0 = 0$, $|\nabla\chi| < \frac{C}{r_{xy}^2}$, we have

$$\left|\int_{\mathscr{E}_R} \vec{w} \cdot \vec{w} \cdot \nabla\chi\, dV\right| < C \int_{\mathscr{E}_R} \frac{|\vec{u}|^2}{r_{xy}^2}\, dV_y.$$

One sees easily that $\vec{w}(x)$, considered as defined only in $\mathscr{E}_R$, can be extended to the entire space (say, by invariance at inverse points with respect to Σ_R) to have Dirichlet Integral not exceeding twice the integral over $\mathscr{E}_R$. Thus,

$$\int_{\mathscr{E}_R} \frac{|\vec{w}|^2}{r_{xy}^2}\, dV \leq C \int_{\mathscr{E}_R} |\nabla\vec{w}|^2\, dV < C R^{-2}$$

by Lemma 4 and Theorem 3. In $\mathscr{S}_R$ we have uniformly $|\nabla\chi| < \frac{C}{R^2}$, hence

$$\left|\int_{\mathscr{S}_R} \vec{w} \cdot \vec{w} \cdot \nabla\chi\, dV\right| < C R\ {}^{2} \int_{\mathscr{E}} |\vec{w}|^2\, dV < C R^{-2},$$

and the lemma follows.

Using Lemma 5, we may complete the proof of Theorem 5. For (17) shows that $\vec{w}(x) = \oint_{\Sigma} \{\ \}\, dS + \vec{w}_{\mathscr{E}}(x)$, where $\vec{w}_{\mathscr{E}}(x)$ has finite square integral over $\mathscr{E}$. This is the decomposition needed for the proof of Theorem 1, which now shows that under the assumptions, $\mathcal{T} = 0$, *q.e.d.*

Theorem 6. *Let* $\vec{w}(x)$ *be a bounded solution of* (2) *defined exterior to a smooth closed surface* Σ, *and suppose* $\vec{w}(x) = 0$ *on* Σ. *Then for every choice of* $\vec{w}_0$, *we shall have* $\int_{\mathscr{E}} |\vec{w} - \vec{w}_0|^2\, dV = \infty$, *or else* $\vec{w}(x) \equiv 0$ *in* $\mathscr{E}$.

Proof. Suppose that for some $\vec{w}_0$, $\int_{\mathscr{E}} |\vec{w} - \vec{w}_0|^2\, dV < \infty$. If $\vec{w}_0 = 0$, then $\vec{w}(x) = \vec{u}(x)$, and hence the constant A of Theorem 3 is zero. Hence, by Theorem 3, $\int_{\mathscr{E}} |\nabla\vec{w}|^2\, dV = 0$, and therefore $\vec{w}(x)$ 0 in $\mathscr{E}$. If $\vec{w}_0 \neq 0$, we conclude first from Theorem 4 that $|\vec{w}(x) - \vec{w}_0| < C r^{-1}$ and then from Theorem 2 that $\int_{\mathscr{E}} |\vec{w} - \vec{w}_0|^2\, dV = \infty$, a contradiction.

★ If $\vec{w}_0 \neq 0$, the best uniform estimate available is $|\nabla\chi| < C r^{-\frac{3}{2}}$, which is not square integrable over $\mathscr{E}$. This is the reason the elaborate apparatus of [2] is necessary in the general case.

Finally, we may obtain a theorem of Liouville type:

Theorem 7. *Let* $\vec{w}(x)$ *be a solution of* (2) *which is defined throughout three-dimensional Euclidean space* $\mathscr{E}$, *and which satisfies the conditions,*

i) $\quad |\vec{w}(x)| < M < \infty \quad$ *throughout* $\mathscr{E}$,

ii) $\quad \int\limits_{\mathscr{E}} |\vec{w}(x) - \vec{w}_0|^2 dV < \infty$

for some constant vector $\vec{w}_0$. *Then* $\vec{w}(x) \equiv \vec{w}_0$ *in* $\mathscr{E}$.

Proof. By Theorem 3, $\int\limits_{\mathscr{E}} |\nabla \vec{w}|^2 dV \equiv 0$, hence $\vec{w}(x) \equiv \text{const}$. By ii), $\vec{w}(x) \equiv \vec{w}_0$, *q.e.d.*

Note added in proof: After submitting this paper I learned that a similar theorem has been proved for motions in two dimensions by W. Wolibner [Studia Math. **12**, 279—285 (1951)], who obtained it as a corollary of a method for determining the force exerted on Σ in the motion. Important extensions of Wolibner's theorems to time-dependent motions of compressible fluids have been given by A. Krzywicki [Studia Math. **15**, 114—122, 174—181, 252—266; **16**, 48—55 (1955—1957)].

References

[*1*] Kotschin, N. J., I. A. Kibel & N. W. Rose: Theoretische Hydromechanik, Bd. II. (Translation from the Russian.) Berlin: Akademie-Verlag 1955.

[*2*] Finn, R.: Estimates at infinity for stationary solutions of the Navier-Stokes equations. (To appear in Bull. Math. Soc. Sci. Math. Phys. R.P.R.)

[*3*] Leray, J.: Étude de diverses équations intégrales non linéaires et de quelques problèmes que pose l'hydrodynamique. J. Math. pure et appl. **9**, 1—82 (1933). See also: Les problèmes non linéaires. L'Enseignement Mathématique **35**, 139—151 (1936).

[*4*] Finn, R.: On steady-state solutions of the Navier-Stokes partial differential equations. Arch. Rational Mech. Anal. **3**, 381—396 (1959).

[*5*] Oseen, C. W.: Neuere Methoden und Ergebnisse in der Hydrodynamik. Leipzig: Akademische Verlagsgesellschaft M. B. H. 1927.

[*6*] Finn, R.: On the steady-state solutions of the Navier-Stokes equations, III. (To appear in Acta Math., Uppsala.)

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Self-Superposable Magnetohydrodynamic Motions

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1. Introduction

The formal concepts of superposability and self-superposability of nonlinear partial differential equations, in particular those equations of interest in hydrodynamics, have been examined in [*1*] and [*2*]. Following a brief review of the literature and some remarks regarding the philosophy of the subject, superposability and self-superposability conditions were established for the equations governing motion of viscous fluids, compressible and incompressible, Newtonian and non-Newtonian. Corresponding conditions were determined for the equations of motion of magnetohydrodynamics and for the potential equation of gas dynamics.

The present paper is an extension of the basic ideas set forth in Chapter 8 of [*2*]. Following these formal results, we now seek solutions of the resultant system of equations, *i.e.*, we wish to construct self-superposable magnetohydrodynamic motions. The appropriate equations are given in the next section. Solutions are obtained in Sections 3 and 4 for two and three-dimensional motions, respectively. Both the cases of steady and nonsteady flow are considered.

The mathematical complexity arises, in part, from the nonlinearity of the basic equations resulting from the terms $(\boldsymbol{U} \cdot \nabla)\, \boldsymbol{U}$ and $(\boldsymbol{B} \cdot \nabla)\, \boldsymbol{B}$ in the momentum equation and curl $(\boldsymbol{U} \times \boldsymbol{B})$ in the induction equation. As a result of this difficulty a large number of the magnetohydrodynamic problems which have been treated with regard to such questions as stability, shock waves, magnetic fields, and aerodynamic applications involved linearizing techniques of one sort or another. In the astronomical literature one finds that very often $\boldsymbol{U}$ is either identically zero or is assumed to be known, in which cases the equations are linear in the remaining electromagnetic variables. Several authors have begun the study of the full equations under simplifying assumptions which make the system more amenable to solution, as is noted in the discussion of Case 2 in Section 4. It is at this point that the introduction of the concept of superposability may be of definite use in exploring the character of the basic equations by providing additional exact solutions and general properties. In general this is our primary purpose, and solution of specific boundary-value problems is not attempted in this work.

Obtaining exact solutions of a particular system of nonlinear differential equations is but one of the potential applications of the concept of superposability. The pertinent conditions would allow one to examine general classes of functions which may have this property of additivity (see [1] and [3], for example). In view of the role that the additivity or superposability property plays in almost all methods for solving linear differential equations, an examination of these methods applied to particular nonlinear equations in the light of generalized superposability results seems warranted. Finally, it is worthwhile to study the several formal mathematical questions which may be raised with regard to a subclass of solutions of a nonlinear equation or system of equations (or, more generally, to nonlinear differential operators of interest) which exhibit this linearity.

2. Magnetohydrodynamic Equations. Superposability and Self-Superposability Conditions

The motion of an electrically conducting fluid in the presence of a magnetic field obeys the equations of magnetohydrodynamics. The fluid is treated as a continuum, and the classical results of fluid dynamics and electrodynamics are combined to express the phenomenon. For a viscous incompressible fluid with constant properties (*i.e.*, σ, μ_e, ε, ϱ, ν) the equations of momentum and of induction constitute the full magnetohydrodynamic system:

$$\begin{gathered}\partial \boldsymbol{U}/\partial t + (\boldsymbol{U}\cdot\nabla)\,\boldsymbol{U} - \lambda(\boldsymbol{B}\cdot\nabla)\,\boldsymbol{B} = -\nabla s + \nu\nabla^2\boldsymbol{U},\\ \lambda = (4\pi\varrho\mu_e)^{-1} = \text{constant}, \qquad s = p/\varrho + \Omega + \lambda\boldsymbol{B}^2/2,\end{gathered} \tag{2.1 a}$$

$$\begin{gathered}\partial \boldsymbol{B}/\partial t = \nabla\times(\boldsymbol{U}\times\boldsymbol{B}) + \eta\nabla^2\boldsymbol{B},\\ \eta = (4\pi\sigma\mu_e)^{-1} = \text{constant},\end{gathered} \tag{2.1 b}$$

where the external force on the fluid other than the Lorentz force is expressed by the potential function Ω, $\boldsymbol{U}$ is the velocity vector, p the pressure, ϱ the density of the fluid, μ_e the magnetic permeability, ε the dielectric constant, ν the kinematic viscosity, σ the electrical conductivity, and $\boldsymbol{B}$ the magnetic induction (the magnetic field strength or intensity is $\boldsymbol{H} = \boldsymbol{B}\mu_e^{-1}$). Inherent in these equations are the following assumptions, often referred to as the magnetohydrodynamic approximations. The electrical component, $\varrho_e\boldsymbol{E}$, of the ponderomotive force in the momentum equation is small compared to the magnetic component, $\boldsymbol{j}\times\boldsymbol{B}$. The displacement current in Ampère's law and the convection current, $\varrho_e\boldsymbol{U}$, in Ohm's law are negligible compared to the conduction current, $\sigma\boldsymbol{E}$. A detailed discussion of these equations and assumptions is available in a number of references ([4]–[6]).

Both $\boldsymbol{U}$ and $\boldsymbol{B}$ are solenoidal vectors; hence

$$\nabla\cdot\boldsymbol{U} = 0, \qquad \nabla\cdot\boldsymbol{B} = 0. \tag{2.1 c, d}$$

System (2.1) must be solved for $\boldsymbol{U}$, $\boldsymbol{B}$, and p. The electric current density, $\boldsymbol{j}$, is then readily obtained from Ampère's law, $\boldsymbol{j} = (4\pi\mu_e)^{-1}\nabla\times\boldsymbol{B}$. The electric field strength, $\boldsymbol{E}$, is obtained from Ohm's law, $\boldsymbol{E} = \boldsymbol{j}\sigma^{-1} - \boldsymbol{U}\times\boldsymbol{B} = \eta\nabla\times\boldsymbol{B} - \boldsymbol{U}\times\boldsymbol{B}$. Clearly $\nabla\cdot\boldsymbol{j} = 0$, while the divergence relation for $\boldsymbol{E}$, $\nabla\cdot\boldsymbol{E} = 4\pi\varepsilon^{-1}\varrho_e$, determines the excess charge density ϱ_e, which need not be zero in general.

The superposability conditions can now be derived in the manner outlined in detail in [1] and [2]. Briefly, if $(\boldsymbol{U}_1, \boldsymbol{B}_1, s_1)$ and $(\boldsymbol{U}_2, \boldsymbol{B}_2, s_2)$ are two solutions of (2.1), what is desired is to determine $s_1+s_2+\pi$ so that $(\boldsymbol{U}_1+\boldsymbol{U}_2, \boldsymbol{B}_1+\boldsymbol{B}_2, s_1+s_2+\pi)$ is also a solution. Introducing the three solutions into (2.1a) and subtracting the latter result from the sum of the first two, one obtains

$$-(\boldsymbol{U}_1\cdot\nabla)\,\boldsymbol{U}_2-(\boldsymbol{U}_2\cdot\nabla)\,\boldsymbol{U}_1+\lambda[(\boldsymbol{B}_1\cdot\nabla)\,\boldsymbol{B}_2+(\boldsymbol{B}_2\cdot\nabla)\,\boldsymbol{B}_1]=\nabla\pi. \tag{2.2a}$$

Similarly, from equation (2.1b), we have

$$-\nabla\times(\boldsymbol{U}_1\times\boldsymbol{B}_2)-\nabla\times(\boldsymbol{U}_2\times\boldsymbol{B}_1)=0. \tag{2.2b}$$

For self-superposable flow, $\boldsymbol{U}_1=\boldsymbol{U}_2=\boldsymbol{U}$, $\boldsymbol{B}_1=\boldsymbol{B}_2=\boldsymbol{B}$ and equations (2.2) become

$$-(\boldsymbol{U}\cdot\nabla)\,\boldsymbol{U}+\lambda(\boldsymbol{B}\cdot\nabla)\,\boldsymbol{B}=\tfrac{1}{2}\Delta\,\pi, \tag{2.3a}$$

$$\nabla\times(\boldsymbol{U}\times\boldsymbol{B})=0, \quad \text{or,} \quad \boldsymbol{U}\times\boldsymbol{B}=\nabla\chi. \tag{2.3b}$$

In order for the system to be self-superposable, *i.e.*, if $(\boldsymbol{U}, \boldsymbol{B})$ is a solution, $(n\,\boldsymbol{U}, n\boldsymbol{B})$ is a solution, there must exist a pair of single-valued scalar functions π and χ defined by (2.3). The conditions of integrability for (2.3) are

$$\operatorname{curl}[(\boldsymbol{U}\cdot\nabla)\,\boldsymbol{U}]=\lambda\operatorname{curl}[(\boldsymbol{B}\cdot\nabla)\,\boldsymbol{B}], \tag{2.4a}$$

$$\operatorname{curl}(\boldsymbol{U}\times\boldsymbol{B})=0. \tag{2.4b}$$

On the assumption that the flow is self-superposable, system (2.1) may be simplified by replacing the nonlinear quantities with π and χ via equations (2.3). Eliminating π from the resultant momentum equation (2.1a) by taking the curl and noting (2.1b), one obtains

$$\partial\boldsymbol{\xi}/\partial t=\nu\,\nabla^2\boldsymbol{\xi}, \qquad \boldsymbol{\xi}=\nabla\times\boldsymbol{U}, \tag{2.5a}$$

$$\partial\boldsymbol{B}/\partial t=\eta\,\nabla^2\boldsymbol{B}. \tag{2.5b}$$

Equations (2.1c, d), (2.4) and (2.5) comprise the self-superposable system which determines $\boldsymbol{U}$ and $\boldsymbol{B}$. The pressure would then be determined from (2.1a). For steady flow (2.5) become

$$\nabla^2\boldsymbol{\xi}=0 \qquad (\nu\neq0), \tag{2.6a}$$

$$\nabla^2\boldsymbol{B}=0 \qquad (\eta\neq0). \tag{2.6b}$$

Similar results were obtained in [2] as well as in [7]. For $\boldsymbol{B}=0$ the conditions and equations noted above reduce to the nonconducting case discussed in [1]. It is of interest to note that for steady flow equation (2.5a) is satisfied for a perfect fluid $(\nu=0)$ and $\boldsymbol{\xi}$ need not be harmonic; thus (2.6a) no longer applies. Similarly, for a steady flow in which the electrical conductivity is infinite $(\eta=0)$ equation (2.5b) is satisfied, and $\boldsymbol{B}$ need not be harmonic.

It may be remarked that the self-superposability condition for equation (2.1a) expressed as equation (8.2.2) of [2] (equations 2.3a or 2.4a above) can be written in the alternate form

$$\operatorname{curl}(\boldsymbol{U}\times\operatorname{curl}\boldsymbol{U})=\lambda\operatorname{curl}(\boldsymbol{B}\times\operatorname{curl}\boldsymbol{B}) \tag{2.7}$$

by using equations (2.1c, d) and the vector identity

$$(\boldsymbol{F}\cdot\nabla)\,\boldsymbol{F}=\tfrac{1}{2}\nabla\boldsymbol{F}^2-\boldsymbol{F}\times\operatorname{curl}\boldsymbol{F}.$$

Force-free fields which are of considerable interest in the study of static equilibrium ($\boldsymbol{U}=0$) in astronomical problems ([*8*] and [*9*]) are characterized by the condition $\boldsymbol{B}\times\operatorname{curl}\boldsymbol{B}=0$. For $\boldsymbol{U}=0$ (2.7) reduces to $\boldsymbol{B}\times\operatorname{curl}\boldsymbol{B}=\nabla\omega$, which contains this result. The self-superposability relation is in fact a generalization of this force-free field condition to the case where $\boldsymbol{U}\neq 0$. Since $\boldsymbol{B}$ satisfies the diffusion equation (2.5b), in the present system there is an obvious analogy with the several electromagnetic and astronomical analyses of the magnetohydrostatic problem, where putting $\boldsymbol{U}=0$ in the induction equation (2.1b) results in this same diffusion equation ([*4*, pp. 4 and 5]; [*13*] and [*14*]).

3. Plane Flow ($\boldsymbol{U}$ independent of z, and $U_z=0$)

Let us assume in this section that the magnetic field is restricted to the plane of motion, *i.e.*, $\boldsymbol{B}$ is independent of z, and $B_z=0$. Introducing the stream function $\psi(x,y,t)$ along with a comparable function $\Phi(x,y,t)$ for the magnetic field so as to satisfy (2.1c, d) identically ($U_x=\psi_{,y}$, $U_y=-\psi_{,x}$; $B_x=\Phi_{,y}$, $B_y=-\Phi_{,x}$ where the comma in the subscript denotes partial differentiation), we reduce equations (2.4, 5) to the form

$$(\nabla_1^2\psi)_{,t}=\nu\nabla_1^4\psi, \tag{3.1a}$$

$$\partial(\psi,\nabla_1^2\psi)/\partial(x,y)=\lambda\,\partial(\Phi,\nabla_1^2\Phi)/\partial(x,y), \tag{3.1b}$$

$$\Phi_{,t}=\eta\nabla_1^2\Phi+f(t), \tag{3.1c}$$

$$\partial(\psi,\Phi)/\partial(x,y)=0. \tag{3.1d}$$

The vorticity ξ, which is normal to the plane of motion ($\xi_x=\xi_y=0$), is $\xi_z=-\psi_{,xx}-\psi_{,yy}=-\nabla_1^2\psi$, where $\nabla_1^4=\nabla_1^2(\nabla_1^2)$. The Jacobian of the functions $h(x,y,t)$ and $g(x,y,t)$ is given by $\partial(h,g)/\partial(x,y)=h_{,x}g_{,y}-h_{,y}g_{,x}$, and $f(t)$ is an arbitrary function of time. Equations (3.1) must be solved for ψ and Φ, while (2.1a) reduces to the following equations for the determination of the pressure, $p/\varrho+\Omega$:

$$(p/\varrho+\Omega)_{,x}=-\psi_{,yt}-\tfrac{1}{2}[\psi_{,x}^2+\psi_{,y}^2]_{,x}+\psi_{,x}\cdot\nabla_1^2\psi-\lambda\Phi_{,x}\cdot\nabla_1^2\Phi+\nu[\nabla_1^2\psi]_{,y}, \tag{3.2a}$$

$$(p/\varrho+\Omega)_{,y}=\psi_{,xt}-\tfrac{1}{2}[\psi_{,x}^2+\psi_{,y}^2]_{,y}+\psi_{,y}\cdot\nabla_1^2\psi-\lambda\Phi_{,y}\cdot\nabla_1^2\Phi-\nu[\nabla_1^2\psi]_{,x}, \tag{3.2b}$$

$$(p/\varrho+\Omega)_{,z}=0. \tag{3.2c}$$

In plane polar coordinates, $\psi(r,\vartheta,t)$ and $\Phi(r,\vartheta,t)$ are defined by the conditions $U_r=r^{-1}\psi_{,\vartheta}$, $U_\vartheta=-\psi_{,r}$; $B_r=r^{-1}\Phi_{,\vartheta}$, $B_\vartheta=-\Phi_{,r}$. Equations (3.1) have the same form with the operator ∇_1^2 now defined by $\nabla_1^2=\partial^2/\partial r^2+r^{-1}\partial/\partial r+r^{-2}\partial^2/\partial\vartheta^2$, and the Jacobian relations now given with respect to r and ϑ. Equations (3.2) apply using this definition for ∇_1^2 and replacing $\partial/\partial x$ by $\partial/\partial r$, $\partial/\partial y$ by $r^{-1}\partial/\partial\vartheta$.

In order to solve system (3.1) for ψ and Φ, let us introduce general orthogonal curvilinear coordinates, α and β, in which case by setting $\alpha=\psi$ the Jacobian relations (3.1b, d) may be simply satisfied. The functions $h_1(\alpha,\beta)$ and $h_2(\alpha,\beta)$ are the usual metric coefficients, $(ds)^2=h_1^2(d\alpha)^2+h_2^2(d\beta)^2$. Equation (3.1d) implies immediately, since $\psi_{,\alpha}=1$ and $\psi_{,\beta}=0$, that

$$\Phi_{,\beta}=0. \tag{3.3a}$$

The vorticity is given by

$$\xi_z=-\nabla_1^2\psi=-(h_1h_2)^{-1}(h_2h_1^{-1})_{,\alpha}, \tag{3.3b}$$

while, by use of (3.3a), $\zeta_z = -\nabla_1^2 \Phi$ becomes

$$\zeta_z = -(h_1 h_2)^{-1} (h_2 h_1^{-1} \Phi_{,\alpha})_{,\alpha} = -h_1^{-2} \Phi_{,\alpha\alpha} + \xi_z \Phi_{,\alpha}. \tag{3.3c}$$

The Jacobian relation (3.1b) reduces to $\xi_{z,\beta} = \lambda \Phi_{,\alpha} \xi_{z,\beta}$, or, by use of (3.3a, c),

$$(1 - \lambda \Phi_{,\alpha}^2)\, \xi_{z,\beta} = -\lambda (h_1^{-2})_{,\beta} \Phi_{,\alpha} \Phi_{,\alpha\alpha}. \tag{3.3d}$$

Using (3.3a—c) in (3.1c) reduces it to the form

$$\Phi_{,t} = \eta (h_1 h_2)^{-1} (h_2 h_1^{-1} \Phi_{,\alpha})_{,\alpha} + f(t) = \eta h_1^{-2} \Phi_{,\alpha\alpha} - \eta \xi_z \Phi_{,\alpha} + f(t). \tag{3.3e}$$

Applying $\partial/\partial\beta$ to (3.3e) and using (3.3a) gives $(h_1^{-2})_{,\beta} \Phi_{,\alpha\alpha} = \xi_{z,\beta} \Phi_{,\alpha}$, and substituting this result into (3.3d) yields

$$\xi_{z,\beta} = 0, \tag{3.3f}$$

from which it follows that

$$(h_1^{-2})_{,\beta} \Phi_{,\alpha\alpha} = 0. \tag{3.4a}$$

By use of (3.3f), (3.1a) becomes

$$\xi_{z,t} = \nu (h_1 h_2)^{-1} (h_2 h_1^{-1} \xi_{z,\alpha})_{,\alpha} = \nu h_1^{-2} \xi_{z,\alpha\alpha} - \nu \xi_z \xi_{z,\alpha}. \tag{3.3g}$$

Applying $\partial/\partial\beta$ to this result and using (3.3f) yields

$$(h_1^{-2})_{,\beta} \xi_{z,\alpha\alpha} = 0. \tag{3.4b}$$

In view of (3.3a) equation (3.1c) would immediately require that $\zeta_{z,\beta} = 0$, in which case (3.3f) follows directly from (3.1b). In addition to being useful in a subsequent application, the somewhat lengthier discussion given above emphasizes the fact that $\xi_{z,\beta} = 0$ is required for all solutions Φ, including, for example, the case where the nonlinear terms are identically zero ($\pi = 0$), *i.e.*, $\Phi = \lambda^{-\frac{1}{2}} \psi$. The significance of (3.3f), namely, that $\xi_z = \xi_z(\alpha, t)$, is quite apparent. First it follows that (3.1b) can be satisfied only if both sides vanish identically, so that $\partial(\psi, \nabla_1^2 \psi)/\partial(x, y) = 0$. (For steady flow $\nabla_1^2 \Phi = \text{constant}$, and (3.1b) reduces immediately to this result.) This implies that the vorticity (in addition to the magnetic field in view of (3.1d)) is constant along the streamlines, where $\psi = \text{constant}$. Finally, equation (3.1a) along with $\partial(\psi, \nabla_1^2 \psi)/\partial(x, y) = 0$ defines $\psi(x, y, t)$ in the corresponding nonconducting case, obtained by KAMPÉ DE FÉRIÉT (see the discussion in [*1*]). To this is now added the related magnetic field for a conducting fluid and the correction to the nonconducting pressure distribution.

All the solutions ψ, Φ of (3.1) may now be obtained from equations (3.3), (3.4). The more interesting ones are discussed in the following outline.

(a) For $\xi_{z,t} \neq 0$ equation (3.4b) is satisfied first by the condition $(h_1^{-2})_{,\beta} = 0$. One is thus led to the class of flows where the streamlines are parallel lines, $\psi = \psi(x, t)$, or concentric circles, $\psi = \psi(r, t)$. In the former case (3.1a) is satisfied by

$$\psi(x, t) = c_1 x^3 + c_2 x^2 + m_1(t)\, x + F(x, t), \tag{3.5}$$

where c_1 and c_2 are arbitrary constants, $m_1(t)$ is an arbitrary function of time, and F is any solution of the one-dimensional heat conduction equation

$$F_{,t} = K_1 F_{,xx} \tag{3.6}$$

for $K_1 = \nu$. It should be remarked at this point that the reduction of our problem to the classical equations of LAPLACE, POISSON, and HELMHOLTZ or to the well known heat conduction or wave equations, *et al.*, completes the solution. In any particular problem the form of the solution will depend on the domain of the independent variables and the initial and boundary conditions to be satisfied, while the general solution may be obtained by separation of variables, the use of the fundamental solution, or the employment of infinite series or various integral techniques. Since these will not be considered here, the reader is referred to the vast literature on such initial value and boundary value problems. Equation (3.1d) shows that Φ is also independent of y (but not necessarily a function of time). The corresponding solutions for Φ from (3.1c) are, therefore,

$$\Phi(x,t) = c_3 x^2 + c_4 x + m_2(t) + F(x,t) \tag{3.5a}$$

where $m_2'(t) - 2C_3\eta = f(t)$ and F is any solution of (3.6) for $K_1 = \eta$, and

$$\Phi(x) = k(2\eta)^{-1} x^2 + c_5 x + c_6, \qquad f(t) = -k. \tag{3.5b}$$

The ($\boldsymbol{i}$) and ($\boldsymbol{k}$) components of equation (2.1a) show that $s_{,x} = s_{,z} = 0$ while the ($\boldsymbol{j}$) component reduces to $(\psi_{,t} - \nu\psi_{,xx})_{,x} = m_1'(t) - 6c_1\nu = s_{,y}$; hence

$$p/\varrho + \Omega = s(y,t) - \tfrac{1}{2}\lambda \boldsymbol{B}^2 = [m_1'(t) - 6c_1\nu]\, y + m_3(t) - \tfrac{1}{2}\lambda\, \Phi_{,x}^2, \tag{3.5c}$$

where $m_3(t)$ is an arbitrary function of time and Φ is given by (3.5a) or (3.5b).

For $\psi = \psi(r,t)$, $\nabla_1^2 = r^{-1}\partial/\partial r(r\,\partial/\partial r)$, and equation (3.1a) is satisfied by

$$\psi(r,t) = [k_1 r^2 + n_1(t)]\ln r + k_2 r^2 + G(r,t) \tag{3.7}$$

where G is any solution of

$$G_{,t} = K_2(G_{,rr} + r^{-1}G_{,r}), \tag{3.8}$$

for $K_2 = \nu$. From (3.1d) it follows that $\Phi_{,\vartheta} = 0$, and the corresponding solutions for Φ are

$$\Phi(r,t) = k_3 \ln r + k_4 r^2 + n_2(t) + G(r,t), \tag{3.7a}$$

where $n_2'(t) - 4\eta k_4 = f(t)$ and G is again a solution of (3.8) ($K_2 = \eta$), and

$$\Phi(r) = k(4\eta)^{-1} r^2 + k_5 \ln r + k_6, \qquad f(t) = -k. \tag{3.7b}$$

Using (2.1a) in polar coordinates, one obtains, first,

$$(\psi_{,t} - \nu\nabla_1^2\psi)_{,r} = r^{-1}[n_1'(t) - 4k_1\nu] = \nu^{-1} s_{,\vartheta}$$

or

$$s(r,\vartheta,t) = [n_1'(t) - 4k_1\nu]\,\vartheta + q(r,t).$$

Applying the result to the e_ϑ-component equation, one obtains

$$s_{,r} = q_{,r} = r^{-1}(\psi_{,r}^2 - \lambda\,\Phi_{,r}^2),$$

and the pressure is therefore given by

$$p/\varrho + \Omega = [n_1'(t) - 4k_1\nu]\,\vartheta + \int r^{-1}(\psi_{,r}^2 - \lambda\,\Phi_{,r}^2)\,dr - \tfrac{1}{2}\lambda\,\Phi_{,r}^2. \tag{3.7c}$$

In view of the fact that the pressure must be uniform in any region, it follows either that $n_1'(t) - 4k_1\nu \neq 0$, in which case the multi-valued potential function Ω

must contain a compensating external force, or, if no such force exists, *i.e.*, if Ω is single-valued, $n_1(t) = 4k_1 \nu t + n_3(t)$. In equation (3.7c) ψ is the solution (3.7), while Φ is given by (3.7a) or (3.7b).

(b) For $\xi_{z,t} \neq 0$ the alternate condition specified by (3.4b) is $\xi_{z,\alpha\alpha} = 0$, *i.e.*, $\xi_z = -\nabla_1^2 \psi$ is a linear function of ψ,

$$\nabla_1^2 \psi = K_3 \psi. \tag{3.9}$$

Equation (3.1a) reduces to

$$\psi_{,t} = \nu \nabla_1^2 \psi, \tag{3.9a}$$

and it follows that

$$\psi(x, y, t) = e^{\nu K_3 t} H(x, y), \tag{3.9b}$$

where H is a solution of

$$\nabla_1^2 H = K_3 H. \tag{3.9c}$$

G. I. TAYLOR [*10*] introduced this solution for the nonconducting problem in his study of the decay of vortices. Corresponding to this solution for ψ, since $(h_1^{-2})_{,\beta} \neq 0$, equation (3.7a) shows that Φ is also linear in ψ, say,

$$\Phi(x, y, t) = m_1(t)\, \psi + m_2(t). \tag{3.9d}$$

Equation (3.1c) then becomes

$$m_1 \psi_{,t} + m_1' \psi + m_2' = \eta\, m_1 \nabla_1^2 \psi + f,$$

or, by use of (3.9),

$$\psi_{,t} = (\eta K_3 - m_1' m_1^{-1})\, \psi + (f - m_2').$$

Comparing this result with (3.9a), one obtains

$$m_1(t) = c\, e^{(\eta-\nu) K_3 t}, \qquad m_2'(t) = f(t),$$

and

$$\Phi(x, y, t) = c\, e^{\eta K_3 t} H(x, y) + m_2(t). \tag{3.9e}$$

Φ decays according to the magnetic viscosity η, while ψ decays according to ν. In both cases the space variation $H(x, y)$ is a solution of (3.9c). If $\eta = \nu = k$, $m_1(t) = \text{constant} = c$, $m_2'(t) = f(t)$, and

$$\Phi = c\, \psi + m_2(t) = c\, e^{k K_3 t} H(x, y) + m_2(t). \tag{3.9f}$$

Generally, electromagnetic dissipation is more significant than viscous dissipation, *i.e.*, $\nu \ll \eta$. Estimates based on a simplified kinetic theory approach for an ionized gas such as hydrogen (reference [5]) show that $\nu/\eta = 2 \times 10^{-7} \alpha/\varrho$, where α is the degree of ionization and ϱ is expressed in cgs units. For sufficiently low values of the density, which will occur in several stellar applications, it is observed that $\nu/\eta \to 1$. From equations (3.2) and (3.9a, d) one readily obtains

$$p/\varrho + \Omega = \tfrac{1}{2}(\psi_{,x}^2 + \psi_{,y}^2) - \tfrac{1}{2} K_3 (\lambda m_1^2 - 1)\, \psi^2 + m_3(t). \tag{3.9g}$$

For $\eta = \nu = k$, $m_1(t) = c$, and if $c = \lambda^{-\frac{1}{2}}$, the nonlinear terms in (2.1a) vanish identically ($\pi = 0$), and (3.9g) reduces to

$$p/\varrho + \Omega = -\tfrac{1}{2}(\psi_{,x}^2 + \psi_{,y}^2) + m_3(t). \tag{3.9h}$$

(c) If $\xi_{z,t}=0$, equation (3.3g) becomes

$$\xi_{z,\alpha\alpha}=h_1^2\,\xi_z\,\xi_{z,\alpha}, \tag{3.10}$$

from which it follows, using (3.4b), that $(h_1^{-2})_{,\beta}\,h_1^2\,\xi_z\,\xi_{z,\alpha}=0$. This is satisfied first by the condition $(h_1^{-2})_{,\beta}=0$, which now refers to the steady flow counterpart of (a), namely $\psi=\psi(x)$ and $\psi=\psi(r)$. For the former, equation (3.1a) yields

$$\psi(x)=c_1\,x^3+c_2\,x^2+c_3\,x+c_4. \tag{3.11}$$

The corresponding solutions for Φ are given by (3.5a, b). Equation (2.1a) now reduces to $s_{,y}=-6c_1\,\nu$, so that $s(x,t)=-6c_1\,\nu\,y+m_1(t)$, and

$$p/\varrho+\Omega=m_1(t)-6c_1\,\nu-\tfrac{1}{2}\,\lambda\,\Phi^2_{,x} \tag{3.11a}$$

gives the pressure distribution corresponding to $\Phi(x,t)$ as given by (3.5a), while if $\Phi=\Phi(x)$, equation (3.5b) may be used, giving

$$p/\varrho+\Omega=c-6c_1\,\nu\,y-\tfrac{1}{2}\,\lambda\,(k\,\eta^{-1}\,x+c_5)^2. \tag{3.11b}$$

Notice that in the last case the well known Poiseuille flow solution given by (3.11) (parabolic velocity distribution) has the usual linear pressure distribution in y modified in this self-superposable example by the magnetic pressure, which appears as a quadratic term in x.

For $\psi=\psi(r)$ equation (3.1a) gives

$$\psi=k_1\,r^2\ln r+k_2\ln r+k_3\,r^2. \tag{3.12}$$

The corresponding expressions for $\Phi(r,t)$ and $\Phi(r)$ are given by equations (3.7a, b). Note that from (3.12) the vorticity is given by $\nabla_1^2\,\psi=4(k_1+k_3)+4k_1\ln r$. If $\Phi(r,t)$ is expressed by equation (3.7a),

$$r^{-1}\,s_{,\vartheta}=-\nu\,(\nabla_1^2\,\psi)_{,r}=-4k_1\,\nu\,r^{-1},\qquad s=-4k_1\,\nu\,\vartheta+q(r,t).$$

$s_{,r}=q_{,r}=r^{-1}(\psi^2_{,r}-\lambda\,\Phi^2_{,r})$, and the pressure is given by

$$p/\varrho+\Omega=-4k_1\,\nu\,\vartheta-\tfrac{1}{2}\,\lambda\,\Phi^2_{,r}+\int r^{-1}(\psi^2_{,r}-\lambda\,\Phi^2_{,r})\,dr. \tag{3.12a}$$

If $\Phi(r)$ is given by (3.7b), this becomes

$$p/\varrho+\Omega=-4k_1\,\nu\,\vartheta-\tfrac{1}{2}\,\lambda\,[k\,(2\eta)^{-1}\,r+k_5\,r^{-1}]^2+\int r^{-1}(\psi^2_{,r}-\lambda\,\Phi^2_{,r})\,dr. \tag{3.12b}$$

By discussion similar to that following (3.7c), we see that Ω must be multi-valued to assure the uniformity of the pressure in (3.12a, b). If this is not so, it is necessary to take $k_1=0$, in which case the solution for $\psi(r)$ reduces to the result $\nabla_1^2\,\psi=4k_3=$constant, which will be considered next.

(d) The second possibility under (c) is $\xi_{z,\alpha}=0$, which says that the vorticity is constant throughout the fluid, *i.e.*,

$$\nabla_1^2\,\psi=c_1. \tag{3.13a}$$

This is a solution of both the steady and unsteady flow problem such that

$$\psi(x,y,t)=a_1\,x^2+a_2\,y^2+b(t)\,x\,y+V(x,y,t), \tag{3.13b}$$

where $b(t)$ is an arbitrary function of time, $2(a_1 + a_2) = c_1$, and V is an arbitrary harmonic function ($\nabla_1^2 V = 0$), while in the steady flow case

$$\psi(x, y) = a_1 x^2 + a_2 y^2 + a_3 x y + V(x, y). \tag{3.13c}$$

Equations (3.3e) and (3.4a) give the corresponding solutions for Φ. From (3.4a) the condition $(h_1^{-2})_{,\beta} = 0$ refers to the previously obtained solutions for $\psi(x, t)$, $\psi(r, t)$, $\psi(x)$, and $\psi(r)$, which now must be modified in view of (3.13a) to give

$$\psi(x, t) = c_1 x^2/2 + m_1(t)\, x + m_2(t), \qquad \psi(r, t) = n_1(t) \ln r + n_2(t), \tag{3.14a, b}$$

$$\psi(x) = c_1 x^2/2 + c_2 x + c_3, \qquad \psi(r) = c_4 \ln r + c_5. \tag{3.14c, d}$$

The solutions for Φ are given by equations (3.5a, b) and (3.7a, b), while the necessary adjustments in (3.5c), (3.7c) and (3.11a, b), (3.12a, b) for the pressure are apparent. If $\Phi_{,t} = 0$, equation (3.3e) becomes $\Phi_{,\alpha\alpha} = (c\eta^{-1} - c_1 \Phi_{,\alpha})\, h_1^2$ where $\xi_z = -\nabla_1^2 \psi = -c_1$ and $f(t) = -c$. Upon substituting this result into (3.4a), we obtain as the only new possibility $\Phi_{,\alpha} = c(c_1 \eta)^{-1}$, or

$$\Phi = c(c_1 \eta)^{-1} + c_6. \tag{3.15}$$

This represents the combination $\nabla_1^2 \psi = c_1$, $\nabla_1^2 \Phi = c\eta^{-1}$ where $\psi(x, y)$ and $\Phi(x, y)$ may be expressed by (3.13c) and (3.15) respectively. Determination of the pressure depends upon specific forms of the arbitrary harmonic function $V(x, y)$. If one specifies $c^2 \lambda = c_1^2 \eta^2$, the nonlinear terms vanish identically ($\pi = 0$).

An alternate assumption for the magnetic field is $\boldsymbol{B} = B_z$. Equation (2.1d) requires that B_z does not depend upon z. In this case $(\boldsymbol{B} \cdot \nabla)\, \boldsymbol{B} \equiv 0$, and system (3.1) applies with Φ replaced by B_z, $f(t) = 0$, and (3.1b) becomes $\partial(\psi, \nabla_1^2 \psi)/\partial(x, y) = 0$. Both the vorticity and the magnetic field are normal to the plane of motion and are constant along streamlines. The solutions for ψ determined above are still applicable, while the corresponding magnetic field (as well as the resultant values of $\boldsymbol{E}$ and $\boldsymbol{j}$) now take on a new physical interpretation. This will have additional significance in subsequent investigations of particular boundary-value problems. Note that an immediate consequence of this assumption is that corresponding to a constant transverse magnetic field, B_z, one obtains the general class of motions wherein the vorticity is everywhere constant or proportional to ψ [equations (3.13) and (3.9)–(3.9c), respectively] in addition to motions corresponding to $\psi(x, t)$, $\psi(r, t)$, $\psi(x)$, and $\psi(r)$, previously defined.

The solutions outlined above for the steady flow case are obviously valid for a perfect fluid as well as one in which the electrical conductivity is infinite. For a perfect fluid ($\nu = 0$) in steady flow, $\psi(x, y)$ need not be biharmonic, as is seen from (3.1a). We require only that $\nabla_1^2 \psi = f(\psi)$, and the equivalence of these self-superposability conditions for plane flow with STOKES condition for ψ to be the stream function for steady flow of a perfect incompressible fluid has been previously noted. For a viscous fluid with infinite electrical conductivity ($\eta = 0$) in steady flow B_z (or Φ, if $f(t) = 0$) need not be harmonic. The stream functions derived above are applicable, while the magnetic field is specified only by the relation B_z (or Φ) $= g(\psi)$. For an inviscid fluid with infinite conductivity, the only requirement in steady flow is that the vorticity and magnetic field be constant along streamlines.

4. Three-dimensional Flow

The assumption of plane flow in Section 3 requires that both conditions $\boldsymbol{U} \neq \boldsymbol{U}(z)$ and $U_z = 0$ be satisfied. R. BERKER [*11*] observed that either restriction may be deleted, resulting in two classes of "pseudo" two-dimensional motions. To study further the character of the present system of equations we shall now examine one of these possibilities with regard to self-superposable magnetohydrodynamic flows. Because of the presence of the magnetic field, this type of motion, which is characterized in detail by BERKER for the nonconducting case, is found to have additional physical significance.

Case 1. $\boldsymbol{U} \neq \boldsymbol{U}(z)$, $U_z \neq 0$.

For the same form of magnetic field, it is easily shown that the potential functions $\psi(x, y, t)$ and $\Phi(x, y, t)$ can be introduced in a manner similar to that of Section 3. The vorticity components, $\boldsymbol{\xi} = \nabla \times \boldsymbol{U}$, and the components of $\boldsymbol{\zeta} = \nabla \times \boldsymbol{B}$ become

$$\boldsymbol{\xi} = \boldsymbol{i}\, U_{z,y} - \boldsymbol{j}\, U_{z,x} - \boldsymbol{k}\, \nabla_1^2 \psi, \qquad \boldsymbol{\zeta} = \boldsymbol{i}\, B_{z,y} - \boldsymbol{j}\, B_{z,x} - \boldsymbol{k}\, \nabla_1^2 \Phi,$$

where ∇_1^2 is the two-dimensional Laplacian operator. The self-superposable system (2.4, 5) can be written as follows:

$$(\nabla_1^2 \psi)_{,t} = \nu \nabla_1^4 \psi, \qquad U_{z,t} = \nu \nabla_1^2 U_z + f_1(t), \tag{4.1 a, b}$$

$$\partial(\psi, \nabla_1^2 \psi)/\partial(x, y) = \lambda\, \partial(\Phi, \nabla_1^2 \Phi)/\partial(x, y), \tag{4.2 a}$$

$$\partial(\psi, U_z)/\partial(x, y) = \lambda\, \partial(\Phi, B_z)/\partial(x, y), \tag{4.2 b}$$

$$\Phi_{,t} = \eta \nabla_1^2 \Phi + f_2(t), \qquad B_{z,t} = \eta \nabla_1^2 B_z, \tag{4.3 a, b}$$

$$\partial(\psi, \Phi)/\partial(x, y) = 0, \qquad \partial(\psi, B_z)/\partial(x, y) = \partial(\Phi, U_z)/\partial(x, y), \tag{4.4 a, b}$$

where $f_1(t)$ and $f_2(t)$ are arbitrary functions of time. Equations (4.1) are obtained from the reduced momentum equation (2.5 a); (4.2) represent the first self-superposability condition (2.4 a); (4.3) are obtained from the reduced induction equation (2.5 b) and (4.4) from the second self-superposability condition (2.4 b). Observe that equations (4.1)–(4.4 a) are just equations (3.1), so that the functions $\psi(x, y, t)$ and $\Phi(x, y, t)$ are the two-dimensional solutions derived in Section 3. Equations (4.1)–(4.4 b) define the corresponding z-components, U_z and B_z, for each two-dimensional solution. The ($\boldsymbol{i}$) and ($\boldsymbol{j}$) component equations of (2.1 a) are unchanged, so that in terms of the obvious $2D$ and $3D$ subscript notation it follows that $s_{3D,x} = s_{2D,x}$ and $s_{3D,y} = s_{2D,y}$. The ($\boldsymbol{k}$) component equation reduces to

$$U_{z,t} + \psi_{,y} U_{z,x} - \psi_{,x} U_{z,y} - \lambda(\Phi_{,y} B_{z,x} - \Phi_{,x} B_{z,y}) = -s_{,z} + \nu \nabla_1^2 U_z.$$

By (4.1 b), (4.2 b), this becomes $s_{,z} = -f_1(t)$ which, along with the results noted above, implies that

$$s_{3D}(x, y, z, t) = -z f_1(t) + s_{2D}(x, y, t) + f_3(t). \tag{4.5 a}$$

Since $s_{3D} = (p/\varrho + \Omega)_{3D} + (\lambda/2)(\Phi_{,x}^2 + \Phi_{,y}^2 + B_z^2)$, it follows that

$$(p/\varrho + \Omega)_{3D} = (p/\varrho + \Omega)_{2D} - (\lambda/2)\, B_z^2 - z f_1(t) + f_3(t), \tag{4.5 b}$$

where $(p/\varrho + \Omega)_{2D}$ is the pressure distribution determined in Section 3.

Introducing α and β in the manner previously described, we obtain from (4.1)–(4.4b) the following additional equations: using (3.3a), from (4.2b) we have

$$U_{z,\beta} = \lambda \Phi_{,\alpha} B_{z,\beta}, \tag{4.6a}$$

and from (4.4b)

$$B_{z,\beta} = \Phi_{,\alpha} U_{z,\beta}. \tag{4.6b}$$

Combining equations (4.6), we obtain $(1 - \lambda \Phi^2_{,\alpha}) U_{z,\beta} = 0$.

(i) $U_{z,\beta} = 0$. This implies $B_{z,\beta} = 0$ from (4.6). Thus U_z and B_z are also functions of ψ, and (4.1, 3b) become

$$U_{z,t} = \nu (h_1 h_2)^{-1} (h_2 h_1^{-1} U_{z,\alpha})_{,\alpha} + f_1(t) = \nu h_1^{-2} U_{z,\alpha\alpha} - \nu \xi_z U_{z,\alpha} + f_1(t), \tag{4.7a}$$

$$B_{z,t} = \eta (h_1 h_2)^{-1} (h_2 h_1^{-1} B_{z,\alpha})_{,\alpha} = \eta h_1^{-2} B_{z,\alpha\alpha} - \eta \xi_z B_{z,\alpha}. \tag{4.7b}$$

Applying $\partial/\partial\beta$ to (4.7) and using (3.6f), we see that

$$(h_1^{-2})_{,\beta} U_{z,\alpha\alpha} = 0, \qquad (h_1^{-2})_{,\beta} B_{z,\alpha\alpha} = 0. \tag{4.8a, b}$$

Following the scheme of Section 3, for a representative number of cases the corresponding solutions for U_z and B_z can now be obtained.

(a) $\xi_{z,t} \neq 0$, $(h_1^{-2})_{,\beta} = 0$. Corresponding to the two-dimensional solutions given by equations (3.5), (3.7) are the following results for U_z and B_z:

$$U_z(x,t) = \boldsymbol{c}_1 x^2 + \boldsymbol{c}_2 x + \boldsymbol{m}_1(t) + F(x,t), \tag{4.9a}$$

where $\boldsymbol{m}_1'(t) - 2\boldsymbol{c}_1 \nu = f_1(t)$, and F is any solution of (3.6) for $k_1 = \nu$.

$$B_z(x,t) = \eta B_{z,xx} \quad \text{or} \quad B_z(x) = \boldsymbol{c}_3 x + \boldsymbol{c}_4 (\eta \neq 0). \tag{4.9b, c}$$

From (3.5c) and (4.5b),

$$p/\varrho + \Omega = [m_1'(t) - 6 c_1 \nu] y + m_3(t) - f_1(t) z - (\lambda/2)(\Phi^2_{,x} + B_z^2), \tag{4.9d}$$

where Φ and B_z are given by (3.5a) or (3.5b), (4.9b), and (4.9c).

$$U_z(r,t) = \boldsymbol{k}_1 \ln r + \boldsymbol{k}_2 r^2 + \boldsymbol{n}_1(t) + G(r,t), \tag{4.10a}$$

where $\boldsymbol{n}_1'(t) - 4\nu \boldsymbol{k}_2 = f_1(t)$, and G is a solution of (3.8) for $k_2 = \nu$.

$$B_z(r,t) = \eta (B_{z,rr} + r^{-1} B_{z,r}) \quad \text{or} \quad B_z(r) = \boldsymbol{k}_3 \ln r + \boldsymbol{k}_4 (\eta \neq 0). \tag{4.10b, c}$$

Equations (3.7c) and (4.5b) provide the corresponding pressure distribution.

(b) $\xi_{z,t} \neq 0$, $(h_1^{-2})_{,\beta} \neq 0$, $\xi_{z,\alpha\alpha} = 0$. Equation (4.8a) indicates that U_z is also linear in ψ,

$$U_z = c_1 \psi, \tag{4.11a}$$

in which case for $f_1(t) = 0$ (4.1b) reduces to (3.9a). The vorticity components are

$$\xi_x = c_1 \psi_{,y} = c_1 U_x, \qquad \xi_y = -c_1 \psi_{,x} = c_1 U_y, \qquad \xi_z = -\nabla_1^2 \psi = -k_3 c_1^{-1} U_z.$$

If $c_1 = \sqrt{-k_3}$, $k_3 < 0$, we note that this solution satisfies the condition

$$\boldsymbol{\xi} = \nabla \times \boldsymbol{U} = c_1 \boldsymbol{U}. \tag{4.11b}$$

The vorticity vector has the same direction as the velocity vector at every point of the fluid and is proportional to it. Motions of this type (Betrami flows) have been treated by several authors [*1*], [*15*]. The solutions for $\psi(x, y, t)$ and $\Phi(x, y, t)$ are given by equations (3.9b, c) and (3.9d–f) respectively. Since B_z is also linear in ψ, it follows that

$$\boldsymbol{\zeta} = \nabla \times \boldsymbol{B} = c_2 \boldsymbol{B}, \tag{4.11c}$$

where the analogy with the previously mentioned force-free field condition of Chandrasekhar is apparent.

The remaining cases are readily derived from the results of Section 3. Let us refer now to the second possible solution.

(ii) $U_{z,\beta} \neq 0$, $\Phi^2_{,\alpha} = \lambda^{-1}$. Thus,

$$\Phi = \lambda^{-\frac{1}{2}} \psi + m_2(t), \tag{4.12a}$$

and from (4.6)

$$U_z = \sqrt{\lambda}\, B_z + g(\alpha, t), \tag{4.12b}$$

i.e., $U_z - \sqrt{\lambda}\, B_z =$ function of ψ. Equations (4.1, 3b) become

$$U_{z,t} = \nu (h_1 h_2)^{-1} [(h_2 h_1^{-1} U_{z,\alpha})_{,\alpha} + (h_1 h_2^{-1} U_{z,\beta})_{,\beta}] + f_1(t), \tag{4.13a}$$

$$B_{z,t} = \eta (h_1 h_2)^{-1} [h_2 h_1^{-1} B_{z,\alpha})_{,\alpha} + (h_1 h_2^{-1} B_{z,\beta})_{,\beta}]. \tag{4.13b}$$

Substituting (4.12b) into (4.13a) and using (4.13b), we obtain

$$\sqrt{\lambda} (\nu\eta)^{-1} (\nu - \eta)\, B_{z,t} + h_1^{-2} g_{,\alpha\alpha} - \xi_z g_{,\alpha} + \nu^{-1} f_1(t) = \nu^{-1} g_{,t}. \tag{4.14}$$

Applying $\partial/\partial\beta$ to (4.14) and using (3.3f) yields

$$\sqrt{\lambda} (\nu\eta)^{-1} (\nu - \eta)\, B_{z,\beta t} + (h_1^{-2})_{,\beta}\, g_{,\alpha\alpha} = 0. \tag{4.15}$$

Representative solutions for U_z and B_z can once again be presented, using the scheme of Section 3.

(a) $\xi_{z,t} \neq 0$, $(h_1^{-2})_{,\beta} = 0$. We shall restrict ourselves to the case $\psi = \psi(x, t)$, $\Phi = \Phi(x, t)$ in the following discussion. Equations (4.1, 3a) become

$$\psi_{,t} = \nu \psi_{,xx} + p_1(t)\, x + p_2(t), \qquad \Phi_{,t} = \eta\, \Phi_{,xx} + f_2(t), \tag{4.16a, b}$$

where p_1 and p_2 are functions of time only and where the Jacobian relations (4.2, 4a) are identically satisfied. We now require, however, the following additional relations:

$$U_z = \sqrt{\lambda}\, B_z + g(x, t), \qquad \Phi = \lambda^{-\frac{1}{2}} \psi + m_2(t). \tag{4.17a, b}$$

Substituting (4.17b) into (4.16b) we have $\psi_{,t} = \eta \psi_{,xx} + \sqrt{\lambda}(f_2 - m_2')$, and comparing this result with (4.16a) yields

$$(\eta - \nu)\, \psi_{,xx} = p_1(t)\, x + p_2(t) - \sqrt{\lambda}\, [f_2(t) - m_2'(t)], \tag{4.18a}$$

or

$$(\eta - \nu)\, \psi_{,t} = \eta\, p_1(t)\, x + \eta\, p_2(t) - \sqrt{\lambda}\, [f_2(t) - m_2'(t)]. \tag{4.18b}$$

Substituting equation (4.17a) into (4.1b) and using (4.3b) gives

$$\sqrt{\lambda} (\eta - \nu)\, \nabla_1^2 B_z = \sqrt{\lambda}\, \eta^{-1} (\eta - \nu)\, B_{z,t} = \nu\, g_{,xx} - g_{,t} + f_1(t). \tag{4.19a}$$

Applying $\partial/\partial y$ to this result, we obtain

$$\sqrt{\lambda}\,\eta^{-1}(\eta-\nu)\,B_{z,yt}=0. \tag{4.19b}$$

Now if $\eta=\nu=k$, equations (4.18) require that $p_1(t)=0$, $p_2(t)=\sqrt{\lambda}\,(f_2-m_2')$, so that $\psi(x,t)$ is now a solution of

$$\psi_{,t}=k\,\psi_{,xx}+p_2(t), \tag{4.20a}$$

and $\Phi=\lambda^{-\frac{1}{2}}\psi+m_2(t)$, where

$$p_2(t)=\sqrt{\lambda}\,(f_2-m_2') \tag{4.20b}$$

defines $\Phi(x,t)$. Equation (4.19b) is satisfied for $B_{z,yt}\neq 0$, and $B_z(x,y,t)$ is thus a solution of the two-dimensional conduction equation,

$$B_{z,t}=k\,\nabla_1^2\,B_z. \tag{4.21a}$$

Equation (4.17a) defines $U_z(x,y,t)$ where $g(x,t)$ is any solution of the one-dimensional equation

$$\nu\,g_{,xx}-g_{,t}+f_1(t)=0. \tag{4.21b}$$

If $\nu\neq\eta$, integrating (4.18a) twice with respect to x, differentiating the result with respect to t, and then comparing with (4.18b), we see that

$$p_1(t)=\text{constant}=a_1,$$

$$p_2(t)-\sqrt{\lambda}\,[f_2(t)-m_2'(t)]=\text{constant}=a_2,$$

$$m_3(t)=a_1\,\eta\,t+a_3,$$

$$m_4(t)=a_2\,\eta\,t+\sqrt{\lambda}\,(\eta-\nu)\,[\textstyle\int f_1(t)\,dt-m_2(t)]+a_4=a_2\,\nu\,t+(\eta-\nu)\textstyle\int p_2(t)\,dt+a_4.$$

Therefore, $\psi(x,t)$ is defined by

$$\psi=\boldsymbol{a}_1\,x^3+\boldsymbol{a}_2\,x^2+m_5(t)\,x+m_6(t). \tag{4.22}$$

This is the special case $\nabla_1^2\,\psi=c_1$ to be considered. The corresponding solution for $\Phi(x,t)$ is given by (4.20b). Equation (4.19b) now requires that $B_{z,yt}=0$; hence

$$B_z=H_1(x,t)+H_2(x,y). \tag{4.23a}$$

Putting (4.23a) into (4.3b), we see that $H_{1,t}-\eta\,H_{1,xx}=\eta\,\nabla_1^2 H_2=H_3(x)$, where H_3 is an arbitrary function of x. Letting $H_1=\boldsymbol{H}_1(x,t)+h_1(x)$ in the equation $H_{1,t}-\eta\,H_{1,xx}=H_3$, we obtain (for $H_3=-\eta\,h_1''$) $\boldsymbol{H}_{1,t}=\eta\,\boldsymbol{H}_{1,xx}$. Similarly, let $H_2=\boldsymbol{H}_2(x,y)+h_2(x)$ in $\eta\,\nabla_1^2 H_2=H_3$, so that for $H_3=\eta\,h_2''$ $(\eta\neq 0)$, we have $\nabla_1^2 H_2=0$. Note that $H_3=\eta\,h_2''=-\eta\,h_1''$ implies that $h_2(x)+h_1(x)=c_1\,x+c_2$. Thus $B_z(x,y,t)$ as defined by (4.23a) becomes

$$H_1=\boldsymbol{H}_1(x,t)+h_1(x), \tag{4.23b}$$

where $\boldsymbol{H}_1$ is a solution of

$$\boldsymbol{H}_{1,t}=\eta\,\boldsymbol{H}_{1,xx}, \tag{4.23c}$$

h_1 is an arbitrary function of x, and

$$H_2=\boldsymbol{H}_2(x,y)+c_1\,x+c_2-h_1(x), \tag{4.23d}$$

where $\boldsymbol{H}_2$ is harmonic, *i.e.*,

$$\nabla_1^2 \boldsymbol{H}_2 = 0. \tag{4.23e}$$

$U_z(x, y, t)$ is now defined by equation (4.17a), where $g(x, t)$ is a solution of

$$\nu g_{,xx} - g_{,t} + f_1(t) = \sqrt{\lambda}\,\eta^{-1}(\eta - \nu)\,\boldsymbol{H}_{1,t}, \tag{4.24}$$

$\boldsymbol{H}_1(x, t)$ having been determined from (4.23c).

(b) $\xi_{z,t} \neq 0$, $(h_1^{-2})_{,\beta} \neq 0$, $\xi_{z,\alpha\alpha} = 0 = \Phi_{,\alpha\alpha}$. The last equation requires that Φ be linear in ψ. In Section 3 equation (3.9a) permitted the required compatibility of the solutions for $\eta \neq \nu$ by proper choice of the arbitraty function $m_1(t)$. It should be noted that the condition $\xi_{z,\alpha\alpha} = 0$, implying that the vorticity be linear in ψ, can be satisfied only by (3.9)—(3.9c) since the choice of K_3 equal to a function of time would ultimately lead to a contradiction. In view of (4.12a) we now require $m_1(t) = \lambda^{-\frac{1}{2}}$ in (3.9d). Substituting this into (4.3a) gives $\psi_{,t} = \eta \nabla_1^2 \psi + \sqrt{\lambda}(f_2 - m_2')$. For $\eta = \nu = k$, $\psi(x, y, t)$ is given by

$$\psi = e^{kK_3 t} H(x, y), \qquad \nabla_1^2 H = K_3 H, \tag{4.25}$$

while (4.12a) defines $\Phi(x, y, t)$ with $m_2'(t) = f_2(t)$. For $\eta \neq \nu$ the only compatible solution possible from $\xi_{z,\alpha\alpha} = 0$ is

$$\nabla_1^2 = c_1. \tag{4.26}$$

Thus, $\psi_{,t} = \eta c_1 + \sqrt{\lambda}(f_2 - m_2')$ and

$$\psi(x, y, t) = m_3(t) + f(x, y), \tag{4.27}$$

where $m_3(t) = \eta c_1 t + \sqrt{\lambda} \int f_2(t)\,dt - \sqrt{\lambda}\, m_2(t)$ and, in view of (4.26), $\nabla_1^2 f = c_1$. The velocity components are thus independent of time. For $\eta = \nu = k$, $B_z(x, y, t)$ is any solution of

$$B_{z,t} = k \nabla_1^2 B_z. \tag{4.28a}$$

Equation (4.12b) defines $U_z(x, y, t)$ where, from (4.15), $g_{,\alpha\alpha} = 0$ implies that $g = n_1(t)\psi + n_2(t)$. In this case $\xi_z = -\nabla_1^2 \psi = -K_3 \psi$, and ψ is given by (4.25). From (4.14), therefore, for $n_2' = f_1$, it follows that $n_1'(t) = 0$, so that

$$g = c_2 \psi + n_2(t), \qquad n_2'(t) = f_1(t). \tag{4.28b}$$

For $\nu \neq \eta$ equations (4.14), (4.15), which define g, show that for $(h_1^{-2})_{,\beta} \neq 0$ it is necessary that $g_{,\alpha\alpha} = 0$ and, further, that $B_{z,\beta t} = 0$. Thus $B_{z,t}$ must be a function of ψ, which merely expresses an integrability condition for g in (4.14); while $B_z(x, y, t)$ is in general a solution of (4.3b), in order that (4.14) define g it is further required that $B_{z,t}$ be a function of ψ. Since $\xi_z = -\nabla_1^2 \psi = -c_1$ is required, equations (4.26)—(4.28) and (4.14) yield, for $n_1(t) = c_2$, the formula

$$\sqrt{\lambda}(y - \eta)^{-1} \eta\, B_{z,t} = c_2 m_3' \qquad (n_2' = c_1 c_2 \nu + f_1).$$

Thus

$$\sqrt{\lambda}(\nu - \eta)^{-1} \eta\, B_z = c_2 m_3(t) + h(x, y),$$

but

$$\sqrt{\lambda}(\nu - \eta)^{-1} \eta \nabla_1^2 B_z = \nabla_1^2 h = \sqrt{\lambda}(\nu - \eta)^{-2} \eta\, B_{z,t} = c_2 \eta^{-1} m_3'.$$

Hence $m_3'(t) = \text{constant} = c_3$, *i.e.*,

$$m_3'(t) = \eta\, c_1 + \sqrt{\lambda}\,[f_2(t) - m_2'(t)] = c_3.$$

As a result of the restrictions imposed on the solution ψ, Φ for $\nu \neq \eta$ by equations (4.12a) and (4.26)–(4.28), B_z is now given by

$$B_z = \eta\, \lambda^{-\frac{1}{2}} (\nu - \eta)^{-1} [c_2 (c_3 t + c_4) + h(x, y)], \qquad \nabla_1^2 h = c_2\, c_3\, \eta^{-1}. \tag{4.28c}$$

(c) $\xi_{z,t} = 0$. For $(h_1^{-2})_{,\beta} = 0$ the solutions corresponding to $\psi(x)$ and $\Phi(x)$ are readily obtained from (a). The second possibility, called (d) in the corresponding discussion of Section 3, is $\zeta_{z,\alpha} = 0$, *i.e.*,

$$\nabla_1^2 \psi = c_1. \tag{4.29}$$

The two-dimensional solutions for ψ and Φ are given by (3.13)–(3.15). In view of (4.12a) we now require that $\nabla_1^2 \Phi = \lambda^{-\frac{1}{2}} \nabla_1^2 \psi = c_1 \lambda^{-\frac{1}{2}}$, *i.e.*, $c^2 \lambda = c_1^2 \eta^2$ in equation (3.15). For $\nu = \eta = k$, g is linear in ψ, which is defined by (4.29) rather than by (4.25). B_z is given by (4.28a) and U_z by (4.12b). For $\nu \neq \eta$, the discussion in (b) applies.

Case 2. $\boldsymbol{U} \neq \boldsymbol{U}(\vartheta)$, $U_\vartheta \neq 0$.

The analogous problem of a "pseudo" two-dimensional motion with axial symmetry is now considered. Magnetohydrodynamic flows of this kind have already been studied somwhat, apart from questions of superposability. The basic equations will therefore be introduced in this section but, for the present, the discussion of the solutions of these equations will be restricted to several immediate results.

Potential functions $\psi(r, z, t)$ and $\Phi(r, z, t)$ can be introduced so as to satisfy the solenoidal condition upon $\boldsymbol{U}(r, z, t)$ and $\boldsymbol{B}(r, z, t)$. Thus

$$\boldsymbol{U} = \boldsymbol{e}_r r^{-1} \psi_{,z} + \boldsymbol{e}_\vartheta U_\vartheta - \boldsymbol{e}_z r^{-1} \psi_{,r}, \qquad \boldsymbol{B} = \boldsymbol{e}_r r^{-1} \Phi_{,z} + \boldsymbol{e}_\vartheta B_\vartheta - \boldsymbol{e}_z r^{-1} \Phi_{,r},$$

and

$$\boldsymbol{\xi} = -\boldsymbol{e}_r U_{\vartheta,z} + \boldsymbol{e}_\vartheta r^{-1} D^2 \psi + \boldsymbol{e}_z r^{-1} (r U_\vartheta)_{,r}.$$

where

$$D^2 = \partial^2/\partial r^2 - r^{-1}\,\partial/\partial r + \partial^2/\partial z^2.$$

System (2.4, 5) becomes

$$(D^2 \psi)_{,t} = \nu D^4 \psi, \qquad (r U_\vartheta)_{,t} = \nu D^2 (r U_\vartheta) + f_1(t), \tag{4.30a, b}$$

$$\partial(\psi, r^{-2} D^2 \psi)/\partial(r, z) + 2 r^{-1} U_\vartheta U_{\vartheta,z} = \lambda [\partial(\Phi, r^{-2} D^2 \Phi)/\partial(r, z) + 2 r^{-1} B_\vartheta B_{\vartheta,z}], \tag{4.30c}$$

$$\partial(\psi, r U_\vartheta)/\partial(r, z) = \lambda\, \partial(\Phi, r B_\vartheta)/\partial(r, z), \tag{4.30d}$$

$$\Phi_{,t} = \eta D^2 \Phi + f_2(t), \qquad (r B_\vartheta)_{,t} = \eta D^2 (r B_\vartheta), \tag{4.30e, f}$$

$$\partial(\psi, \Phi)/\partial(r, z) = 0, \qquad \partial(\psi, r^{-1} B_\vartheta)/\partial(r, z) = \partial(\Phi, r^{-1} U_\vartheta)/\partial(r, z). \tag{4.30g, h}$$

For the magnetohydrostatic problem ($\boldsymbol{U} = 0$) CHANDRASEKHAR ([8] and [9]) examined the case of a force-free field which, in general, is characterized by the equation $\boldsymbol{B} \times \operatorname{curl} \boldsymbol{B} = 0$. As previously observed, this relation is a specialization of the general self-superposability equation derived earlier. It is examined in some detail for the axially symmetric problem presently being considered, and the reader is referred to these papers for the solutions when $\boldsymbol{U}=0$. A non-zero velocity is accounted for in [*12*], where, under the additional restriction that the fluid be inviscid, a comprehensive treatment of the full axisymmetric equations is given. Included are several interesting solutions of the full system, the derivation of a number of general expressions and integral relations, and a discussion of boundary condition considerations. In view of the obvious need for examining system (4.30) in its own right a detailed analysis will not be attempted at this time. Simplifications for the steady flow case are apparent, while some rather general remarks relevant to the corresponding two-dimensional case ($U_\vartheta=0=B_\vartheta$) and to the immediate solution $\boldsymbol{U}=\sqrt{\lambda}\boldsymbol{B}$ (note [*12*]) are better left for a detailed analysis. For the purpose of illustration, however, let us consider the following particular examples.

(i) $\Phi=\Phi(r)$. Equation (4.30g) shows that $\psi=\psi(r)$. The operator D^2 now assumes the form $D^2=r\,\partial/\partial r\,(r^{-1}\partial/\partial r)$, so that [for $\nu=0$, (4.30a) is identically satisfied and the following is not required] $D^4\psi=0$ provides, after several integrations, the well known Poiseuille motion:

$$\psi=c_1 r^2\ln r+c_2 r^4+c_3 r^2, \tag{4.31 a}$$

where the c_i are arbitrary constants. From (4.30e), taking $k_1=-f_2(t)$, we have $D^2\Phi=k_1\eta^{-1}$, so that

$$\Phi=k_1(2\eta)^{-1}r^2\ln r+c_4 r^2. \tag{4.31 b}$$

From (4.30d), $\psi_{,r}U_{\vartheta,z}=\lambda\Phi_{,r}\beta_{\vartheta,z}$ while, from (4.30h), $\psi_{,r}B_{\vartheta,z}=\Phi_{,r}U_{\vartheta,z}$. It follows that

$$\psi_{,r}\Phi_{,r}^{-1}=\lambda B_{\vartheta,z}U_{\vartheta,z}^{-1}=U_{\vartheta,z}B_{\vartheta,z}^{-1}.$$

Clearly, there are two possibilities involved, namely, $U_{\vartheta,z}=B_{\vartheta,z}=0$ or $U_{\vartheta,z}=\sqrt{\lambda}B_{\vartheta,z}$. In the former case, all the Jacobian relations (4.30c, d, g, h) are identically satisfied, and complementing equations (4.31a, b) are similar expressions for rU_ϑ and rB_ϑ as obtained from the relations $D^2(rU_\vartheta)=k_2\nu^{-1}\left(k_2=-f_2(t)\right)$, $D^2(rB_\vartheta)=0$ ($\eta\neq 0$) using $D^2=r\,\partial/\partial r\,(r^{-1}\partial/\partial r)$. Alternatively, $U_{\vartheta,z}=\sqrt{\lambda}B_{\vartheta,z}$ and, therefore, $\psi_{,r}=\sqrt{\lambda}\Phi_{,r}$. From the latter it follows that $\psi=\sqrt{\lambda}\Phi+\text{constant}$, whence it is now required that $c_1=k_1\sqrt{\lambda}(2\eta)^{-1}$, $c_2=0$, $c_3=\sqrt{\lambda}c_4$. In addition,

$$U_\vartheta=\sqrt{\lambda}B_\vartheta+g(r). \tag{4.32}$$

Substituting (4.32) into $D^2(rU_\vartheta)=k_2\nu^{-1}$, we obtain

$$D^2\left(r\sqrt{\lambda}B_\vartheta+rg\right)=\sqrt{\lambda}D^2(rB_\vartheta)+D^2(rg)=D^2(rg)=k_2\nu^{-1},$$

since, from (4.30f) for $\eta\neq 0$, $D^2(rB_\vartheta)=0$. Substituting (4.32) into (4.30c) yields

$$\begin{aligned}2r^{-1}U_\vartheta U_{\vartheta,z}&=2r^{-1}\left(\sqrt{\lambda}B_\vartheta+g\right)\sqrt{\lambda}B_{\vartheta,z}\\&=2\lambda r^{-1}B_\vartheta B_{\vartheta,z}+2\sqrt{\lambda}r^{-1}g B_{\vartheta,z}=2\lambda r^{-1}B_\vartheta B_{\vartheta,z}.\end{aligned}$$

Since $B_{\vartheta,z}\neq 0$ for this case, we must have $g\equiv 0$ (and therefore $k_2=0$), so that $U_\vartheta=\sqrt{\lambda}B_\vartheta$ where $B_\vartheta(r,z)$ is a solution of $D^2(rB_\vartheta)=0$.

(ii) $\Phi=\Phi(r, t)$. The previous discussion pertaining to the equations containing Jacobians still applies. We require, therefore, that $\psi=\psi(r, t)$ and that either $U_{\vartheta, z}=B_{\vartheta, z}=0$, in which case all the Jacobian relations are identically satisfied, or $U_{\vartheta, z}\neq 0$, $B_{\vartheta, z}\neq 0$. In the former case the solutions for ψ, Φ, U_ϑ and B_ϑ are readily obtained from (4.30a, b, e, f), using $D^2=r\,\partial/\partial r\,(r^{-1}\,\partial/\partial r)$ to express the resultant one-dimensional nonsteady flow equations. For $U_{\vartheta, z}\neq 0$, $B_{\vartheta, z}\neq 0$ we obtain once more $U_{\vartheta, z}=\sqrt{\lambda}\,B_{\vartheta, z}$ and $\psi_{,r}=\sqrt{\lambda}\,\Phi_{,r}$. From the latter, $\psi=\sqrt{\lambda}\,\Phi-h(t)$, and the solution for ψ and Φ from (4.30a) and (4.30e) is thus restricted in a manner already experienced, *i.e.*, depending on the constants η and ν. In addition, $U_\vartheta=\sqrt{\lambda}\,B_\vartheta+g(r, t)$ is specified by (4.30c) in that $g=0$ is required for $B_{\vartheta, z}\neq 0$. Referring to equations (4.30b, f), a consistant solution is obtained only when $\nu=\eta$ and $f_1(t)=0$.

References

[*1*] Gold, Richard R., & M. Z. v. Krzywoblocki: On superposability and self-superposability conditions for hydrodynamic equations based on continuum. I. J. reine angew. Mathematik **199**, H. 3/4, 139—164 (1958).

[*2*] Gold, Richard R., & M. Z. v. Krzywoblocki: On superposability and self-superposability conditions for hydrodynamic equations based on continuum. II. J. reine angew. Mathematik **200**, H. 3/4, 140—169 (1958).

[*3*] Strang, J. A.: Superposable fluid motions. Ankara Université, Faculté des Sciences, Comm. **1**, 1—32 (1948).

[*4*] Cowling, T. G.: Magnetohydrodynamics. New York: Interscience Publishers, Inc. 1957.

[*5*] Elsasser, W. M.: Hydromagnetic dynamo theory. Rev. of Modern Physics **28**, No. 2, 135—136 (1956).

[*6*] Sears, W. R.: Magnetohydrodynamic effects in aerodynamic flows. J. Aero-Space Sci. **29**, No. 6, 397—406 (1959).

[*7*] Kapur, J. N.: Superposability in Magnetohydrodynamics. Appl. Sci. Res., Sect. A **8**.

[*8*] Chandrasekhar, S.: On force-free magnetic fields. Proc. National Acad. Sci. **42**, No. 1, 1 (1956).

[*9*] Chandrasekhar, S., & K. Prendergast: The equilibrium of magnetic stars. Proc. National Acad. Sci. **42**, No. 1, 5 (1956).

[*10*] Taylor, G. I.: On the decay of vortices in a viscous fluid. Phil. Mag. **46**, 671—674 (1923).

[*11*] Ratib Berker: Sur Quelques Cas d'Intégration des Équations du Mouvement d'un Fluide Visqueux Incompressible. Thesis, University of Lille, France, 1936.

[*12*] Chandrasekhar, S.: Axisymmetric magnetic fields and fluid motions. Astrophysical J. **124**, No. 1, 232—243 (1956).

[*13*] *ibid.* pp. 244—265.

[*14*] Pao, S.: Astrophysical J. **124**, pp. 266—271.

[*15*] Truesdell, C., & R. Toupin: The Classical Field Theories. Handbuch der Physik, Vol. 3, Part I. Berlin-Göttingen-Heidelberg: Springer 1960. See §§ App. 34, 112, 114.

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Monotoniesätze bei Differentialgleichungen 2. Ordnung

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§ 1. Randwertaufgaben bei gewöhnlichen und partiellen Differentialgleichungen

B sei ein beschränktes offenes Gebiet des n-dimensionalen Raumes der Veränderlichen $(x_1, x_2, \ldots, x_n) = (x_i)$ mit dem Rand Γ und $\overline{B} = B + \Gamma$. $u(x_i) \in K$ soll heißen: u besitzt in $\overline{B}$ stetige Ableitungen bis zur 1. und in B stetige Ableitungen bis zur 2. Ordnung einschließlich. Zur Abkürzung setzen wir $u_j = \frac{\partial u}{\partial x_j}$ und $u_{jk} = \frac{\partial^2 u}{\partial x_j \, \partial x_k}$. In B sei der Operator

$$(1.1) \quad Tu = -a_{jk} \cdot u_{jk} + F(x_i, u, u_j) \text{ und auf } \Gamma\text{: } U_\mu u = u - \alpha \frac{\partial u}{\partial \sigma} \quad (\mu = 1, 2, \ldots, m)$$

definiert. Dabei wurde über doppelt auftretende Indizes von 1 bis n summiert. Die symmetrische Matrix $a_{jk}(x_i) = a_{kj}(x_i)$* sei für jeden Punkt aus B positiv definit, $\alpha \geqq 0$ auf Γ und $\partial/\partial\sigma$ die Ableitung auf dem Rande in Richtung der Konormalen (s. Collatz [*1*], S. 23). Dann ist bekannt (Collatz [*1*] und [*2*], Meyer [*3*]), daß unter der Voraussetzung: $F(x_i, u, u_j)$ monoton nicht-fallend hinsichtlich u in einem Bereich S des (x_i, u, u_j)-Raumes mit $(x_i) \in B$ der Monotoniesatz gilt, d.h. für zwei Funktionen $u(x_i)$, $v(x_i) \in K$ folgt aus $Tu < Tw$ in B und $U_\mu u \leqq U_\mu v$ auf Γ stets $u \leqq v$ in $\overline{B}$. Die Funktion F war dabei als stetige Funktion ihrer Argumente vorausgesetzt worden.

In diesem Paragraphen soll gezeigt werden, daß der Monotoniesatz auch für gewisse hinsichtlich u monoton fallende Funktionen F Gültigkeit behält. Dabei wird der Beweis des bisherigen Monotoniesatzes als bekannt vorausgesetzt, so daß wir uns an vielen Stellen im folgenden kurz fassen können.

Wir treffen folgende *Voraussetzungen:*

(I) $F(x_i, u, u_j)$ besitze in S stetige partielle Ableitungen nach u und u_j.

(II) Für fest gewählte stetige Funktionen $B_j(x_i)$* und $C(x_i)$ existiere eine Funktion $\varphi(x_i) \in K$ mit

$$(1.2) \quad \varphi > 0 \text{ in } B, \quad L^*\varphi = -a_{jk} \cdot \varphi_{jk} + B_j \cdot \varphi_j + C \cdot \varphi \geqq 0 \text{ in } B, \quad U_\mu \varphi > 0 \text{ auf } \Gamma.$$

(III) Es sei die Funktion

$$(1.3) \quad G(x_i, z) = F(x_i, v + \varphi z, v_j + \varphi_j z) - F(x_i, v, v_j) - B_j \cdot \varphi_j \cdot z - C \cdot \varphi \cdot z \geqq 0$$

für $z \geqq 0$ mit den in (II) gewählten Funktionen φ, B_j, C und eine fest gewählte Funktion $v(x_i) \in K$.

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* Hier dienen die Indizes nur zur Unterscheidung der Funktionen.

Satz 1. *Unter den Voraussetzungen* (I) *bis* (III) *und* $Tu < Tv$ *in* B, $U_\mu u \leqq U_\mu v$ *auf* Γ *für zwei Funktionen* $u, v \in K$, *wobei* v *die in* (1.3) *gewählte Funktion bedeutet, gilt* $u \leqq v$ *in* $\bar{B}$.

Beweis. Für die Funktion $z = \frac{u - v}{\varphi}$ definieren wir den linearen Operator

$$\tilde{L} z = -\varphi \cdot a_{jk} \cdot z_{jk} - 2 \cdot a_{jk} \cdot \varphi_j \cdot z_k + L^* \varphi \cdot z \quad \text{in } B. \tag{1.4}$$

Andererseits erhalten wir mit $u = v + \varphi \cdot z$, $u_j = v_j + \varphi \cdot z_j + \varphi_j \cdot z$ und

$$\begin{aligned} -a_{jk}(u - v)_{jk} &= -\varphi \cdot a_{jk} \cdot z_{jk} - 2 \cdot a_{jk} \cdot \varphi_j \cdot z_k - a_{jk} \cdot \varphi_{j..} \cdot z \\ &= Tu - Tv - F(x_i, u, u_j) + F(x_i, v, v_j) \end{aligned}$$

$$\begin{aligned} \tilde{L} z = Tu - Tv - F(x_i, v + \varphi z, v_j + \varphi z_j + \varphi_j z) + \\ + F(x_i, v, v_j) + (B_j \cdot \varphi_j + C \cdot \varphi) \cdot z. \end{aligned} \tag{1.5}$$

Auf Γ wird $U_\mu(u - v) = U_\mu \varphi \cdot z - \alpha \varphi \frac{\partial z}{\partial \sigma} \leqq 0$. Sei nun $\tilde{z} = \operatorname{Max}_{\bar{B}} z(x_i)$, das im Punkte $P_0 = (x_{0i})$ angenommen werden möge. Ist $\tilde{z} \le 0$, so ist der Satz 1 richtig. Nehmen wir also an, es gelte $\tilde{z} > 0$. Dann kann P_0 nicht auf Γ liegen, denn dort wäre $\left(\frac{\partial z}{\partial \sigma}\right)_{P_0} \leqq 0$ und somit $(z)_{P_0} \leqq \left(\frac{\alpha \cdot \varphi}{U_\mu \varphi} \cdot \frac{\partial z}{\partial \sigma}\right)_{P_0} \leqq 0$ auf Grund der Voraussetzungen über φ und α. Für $P_0 \in B$ besitzt z ein differenzierbares Maximum mit $(z_j)_{P_0} = 0$, d.h. $(u_j)_{P_0} = (v_j + \varphi_j \cdot z)_{P_0}$ und $-(a_{jk} \cdot z_{jk})_{P_0} \geqq 0$. Dann gilt für den Operator $\tilde{L} z$ nach (1.4) mit $\tilde{z} > 0$ und der Voraussetzung (1.2): $(\tilde{L} z)_{P_0} \geqq 0$ und nach (1.5) mit $Tu - Tv < 0$ in B und der durch (1.3) eingeführten Funktion: $(\tilde{L} z)_{P_0} < -G(x_{0i}, \tilde{z}) \leqq 0$ wegen $\tilde{z} > 0$. Diese beiden Ergebnisse widersprechen sich, d.h. die Annahme $\tilde{z} > 0$ ist falsch, es gilt $z \leqq \tilde{z} \leqq 0$ in $\bar{B}$.

Wie aus dem Beweisgang hervorgeht, gilt der

Zusatz. *Der Satz 1 bleibt unter der Voraussetzung* $Tu \le Tv$ *in* B, $U_\mu u \le U_\mu v$ *auf* Γ *richtig, wenn statt* (1.2)

$$\varphi > 0 \text{ und } L^* \varphi > 0 \text{ in } B, \quad U_\mu \varphi > 0 \text{ auf } \Gamma \tag{1.2*}$$

oder statt (1.3)

$$G(x_i, z) > 0 \quad \text{für} \quad z > 0 \tag{1.3*}$$

gefordert wird.

Die Hinzunahme des Gleichheitszeichens in $Tu \leqq Tv$ in B kann bei Erfüllung von (1.2) und (1.3) auch unter gewissen Voraussetzungen über die Funktion $F(x_i, u, u_j)$ bewiesen werden (s. z.B. den Zusatz zum Satz 3), doch soll dieses hier nicht weiter durchgeführt werden, da man in den meisten Fällen mit (1.2*) oder (1.3*) auskommen dürfte.

Bemerkungen zum Satz 1. Die Anwendbarkeit des Monotoniesatzes hängt davon ab, ob es gelingt, Funktionen φ, B_j, C so zu bestimmen, daß die Voraussetzungen (II) und (III) erfüllt sind. Für einige Sonderfälle soll dieses im folgenden untersucht werden.

1. Sei $F_u(x_i, u, u_j) \geqq 0$ in dem hinsichtlich u konvexen Bereich S, so lassen sich die Voraussetzungen (II) und (III) mit $\varphi = 1$, $B_j = C = 0$ erfüllen, denn

dann wird $G(x_i, z) = F(x_i, v+z, v_j) - F(x_i, v, v_j) = F_u(x_i, \tilde{v}, v_j) \cdot z \geqq 0$ für $z \geqq 0$ ($\tilde{v}$ bedeutet hier eine Zwischenstelle zwischen v und $v+z$). Wir erhalten also den Monotoniesatz in der bekannten Fassung.

2. Für lineares $F = -b_j \cdot u_j + c \cdot u$ (wobei der Fall $c \geqq 0$ bereits durch 1. erledigt ist, wir lassen also auch negative c-Werte zu) wählen wir $B_j = -b_j$ und $C = c$ und erhalten damit $G(x_i, z) = 0$. Wenn sich zeigen läßt, daß irgendeine Funktion $\varphi(x_i)$ mit den Eigenschaften $-a_{jk} \cdot \varphi_{jk} - b_j \cdot \varphi_j + c \cdot \varphi \geqq 0$ und $\varphi > 0$ in B, $U_\mu \varphi > 0$ auf Γ existiert, so läßt sich der Satz 1 anwenden. In vielen Fällen gelingt es sehr einfach, solch eine Funktion φ anzugeben (s. hierzu die Beispiele). Besitzt die Eigenwertaufgabe $-a_{jk} \cdot w_{jk} - b_j \cdot w_j + \lambda \cdot c \cdot w = 0$ in B und $U_\mu w = 0$ auf Γ den kleinsten Eigenwert λ_1 und ist $\lambda_1 > 1$, so gibt es eine zugehörige Eigenfunktion $w_1(x_i) > 0$ in B. Diese Funktion erfüllt dann im wesentlichen (man muß wegen der Forderung $U_\mu \varphi > 0$ auf Γ noch eine Grenzbetrachtung durchführen) die Voraussetzung (1.2). Aber der Nachweis $\lambda_1 > 1$ wird wohl häufig nicht einfacher zu erbringen sein, als die direkte Konstruktion einer Funktion φ.

3. Für die in u nicht-lineare Funktion $F = -b_j \cdot u_j + f(x_i, u)$ setzen wir $B_j = -b_j$ und $C = F_u(x_i, v) = f_u(x_i, v)$ und erhalten

$$G(x_i, z) = f(x_i, v + \varphi z) - f(x_i, v) - f_u(x_i, v) \cdot \varphi \cdot z = \tfrac{1}{2} f_{uu}(x_i, \tilde{v}) \cdot (\varphi z)^2,$$

wenn wir die 2. partielle Ableitung von f nach u in einem bezüglich u konvexen Bereich S voraussetzen. Ist dort $F_{uu} = f_{uu} \geqq 0$, so wird die Voraussetzung (III) $G(x_i, z) \geqq 0$ für $z \geqq 0$ erfüllt. Zum Beispiel wird für $f = -u^3$ stets $f_u \leqq 0$, der Monotoniesatz in der bisherigen Fassung wäre nicht anwendbar. Gilt jedoch $u \leqq 0$ in S, so ist $f_{uu} \geqq 0$. Die Voraussetzung (II) verlangt

$$-a_{jk} \cdot \varphi_{jk} - b_j \cdot \varphi_j + F_u(x_i, v) \cdot \varphi \geqq 0 \tag{1.6}$$

in B mit $U_\mu \varphi > 0$ auf Γ für eine in B positive Funktion φ. Zweckmäßig wird man hier eine Minorante bestimmen: $F_u(x_i, u) \geqq -\omega^2 = \text{const}$ und statt (1.6)

$$-a_{jk} \cdot \varphi_{jk} - b_j \cdot \varphi_j - \omega^2 \cdot \varphi \geqq 0 \tag{1.6'}$$

in B untersuchen.

4. Ist u eine Funktion der einen Veränderlichen $x_1 = x$, so haben wir den Operator

$$Tu = -u'' + F(x, u, u') \quad \text{für} \quad a_1 < x < a_2 \qquad \left({}' = \frac{d}{dx}\right) \tag{1.7}$$

und die linearen Randbedingungen

$$U_\mu u = u(a_\mu) + (-1)^\mu \alpha_\mu \cdot u'(a_\mu) \quad \text{mit} \quad \alpha_\mu \geqq 0 \qquad (\mu = 1, 2). \tag{1.8}$$

Die folgenden einfachen Beispiele mögen zur Erläuterung des Satzes dienen. Bei der Durchrechnung kam es nicht auf sehr große Genauigkeit an — die ließe sich leicht durch Hinzunahme weiterer Glieder in den Ansätzen steigern —, es sollte vor allen Dingen die Einfachheit des Verfahrens gezeigt werden. Alle Beispiele lassen sich nach dem bisherigen Monotoniesatz nicht behandeln, man war bei ihnen auf andere Methoden angewiesen, z.B. Iterationsverfahren, Ritzsches Verfahren usw., bei denen Fehlerabschätzungen mühevoller sind und meistens auch schlechter ausfallen.

Beispiel 1. Lineare gewöhnliche Differentialgleichung (s. COLLATZ [*1*], S. 183)

$$(1.9)\qquad Tu = -u'' - (1+x^2)\,u = 1 \quad \text{für} \quad |x| < 1, \qquad u(\pm 1) = 0.$$

Hier ist $c = -(1+x^2) < 0$, um den Satz 1 anwenden zu können, müssen wir zunächst die Existenz einer Funktion $\varphi(x) > 0$ mit den Eigenschaften $L^*\varphi = -\varphi'' - (1+x^2)\,\varphi \geqq 0$ für $|x| < 1$ und $\varphi(\pm 1) > 0$ nachweisen. Für $\varphi = 1 - \frac{2}{3}\cdot x^2$ wird $L^*\varphi = \frac{1}{3}(1 - x^2 + 2x^4) > 0$ (die Funktion $\varphi = \cos\sqrt{2}\,x$ würde ebenfalls die verlangten Forderungen erfüllen). Der Satz 1 kann damit auf den Operator Tu aus (1.9) angewendet werden. — Als Näherungslösung für u machen wir einen Potenzreihenansatz, der die Randbedingungen bereits erfüllt:

$$v(x) = a + b\,x^2 + c\,x^4 - (a + b + c)\,x^6.$$

Es wird dann

$$\begin{aligned} Tu - Tv = {} & (1 + a + 2b) + (a + b + 12c)\,x^2 - \\ & - (30a + 29b + 29c)\,x^4 - (a+b)\,x^6 - (a + b + c)\,x^8. \end{aligned}$$

Aus der Forderung, daß die ersten Potenzen bis x^4 einschließlich verschwinden sollen, folgt eine 1. Näherung $v_1 = \frac{1}{343}(319 - 331\,x^2 + x^4 - 11\,x^6)$ mit $Tu - Tv_1 = \frac{x^6}{343}(12 + 11x^2) \geqq 0$, also $v_1 \leqq u$ für $|x| \leqq 1$. Aus der Forderung $Tu - Tv = 0$ für $x = 0$ und $x = 1$, $(Tu - Tv)'' = 0$ für $x = 0$ ergibt sich eine 2. Näherung $v_2 = \frac{1}{366}(342 - 354\,x^2 + x^4 - 11\,x^6)$ mit $Tu - Tv_2 = -\frac{x^4}{366}(23 - 12\,x^2 - 11\,x^4) \leqq 0$, also $u \leqq v_2$ für $|x| \leqq 1$. Insgesamt erhalten wir somit die Schranken

$$(1.10)\qquad \begin{aligned} \frac{1}{343}(319 - 331x^2 + x^4 - 11x^6) \leqq u(x) \leqq \frac{1}{366}(342 - 354\,x^2 + x^4 - 11\,x^6) \\ \text{für } |x| \leqq 1\,. \end{aligned}$$

Zum Beispiel für $x = 0$: $0{,}9300 \leqq u(0) \leqq 0{,}9345$.

Beispiel 2. Nichtlineare gewöhnliche Differentialgleichung

$$(1.11)\qquad Tu = -u'' - u - \tfrac{1}{3}u^2 = \tfrac{1}{2} \quad \text{für} \quad |x| < 1, \qquad u(\pm 1) = 0.$$

Den Bereich S des (x, u)-Raumes legen wir fest durch $|x| \leqq 1$, $0 \leqq u \leqq U = \text{const.}$ Dann wird in S: $F_u = -1 - \frac{2}{3}u \geqq -1 - \frac{2}{3}U = -\omega^2$ und $\varphi(x) > 0$ ist nach (1.6′) aus der Bedingung $L^*\varphi = -\varphi'' - \omega^2\varphi > 0$ zu bestimmen. Für $\varphi = 1 - \tau\cdot x^2$ mit $\tau < 1$ wird $L^*\varphi = 2\tau - \omega^2(1 - \tau x^2) \geqq 2\tau - \omega^2 > 0$ für $\omega^2 < 2\tau < 2$, d.h. der Satz 1 ist anwendbar, wenn $u \leqq U < \frac{3}{2}$ in S bleibt. Diese Bedingung muß hinterher aus den errechneten Schranken überprüft werden. Der Ansatz $v = a(1 - x^2)$ liefert $Tv = a\left(1 + x^2 - \frac{a}{3}(1 - x^2)^2\right)$. Die Forderung $Tv \leqq Tu = \frac{1}{2}$ liefert den Wert $a_1 = \frac{1}{4}$ und $Tv \geqq Tu = \frac{1}{2}$ den Wert $a_2 = 0{,}6340$. Somit haben wir also zunächst die groben Schranken

$$0{,}25\,(1 - x^2) \leqq u(x) \leqq 0{,}6340\,(1 - x^2) \leqq 0{,}6340 = U < \tfrac{3}{2}.$$

Um bessere Schranken zu erhalten, setzen wir $v = (a + b\,x^2)\,(1 - x^2)$. Verlangen wir $Tu - Tv = 0$ für $x = 0$ und $x = 1$, so erhalten wir $a_1 = 0{,}4845$, $b_1 = -0{,}0469$ und mit diesen Werten $Tu - Tv \leqq 0$. Aus $Tu - Tv = (Tu - Tv)'' = 0$ für $x = 0$

folgen $a_2 = 0{,}4580$, $b_2 = -0{,}0560$ und $Tu - Tv \geqq 0$, also insgesamt die Schranken

$$(0{,}4580 - 0{,}0560\,x^2)(1 - x^2) \leq u(x) \leqq (0{,}4845 - 0{,}0469\,x^2)(1 - x^2) \quad \text{für} \quad |x| \leqq 1.$$

Bei Hinzunahme weiterer Glieder steigt der Rechenaufwand außerordentlich rasch an, da dann nichtlineare Gleichungssysteme zu lösen sind.

Bei gewöhnlichen Differentialgleichungen wird man im allgemeinen die Näherungslösung $v(x)$ so wählen, daß die Randbedingungen erfüllt sind. Im Gegensatz dazu ist es bei partiellen Differentialgleichungen häufig möglich, von Näherungslösungen auszugehen, die die Differentialgleichung exakt erfüllen. Bei den linearen Differentialgleichungen gelingt es, eine einfache Fehlerabschätzung aus den Defekten auf dem Rande anzugeben. Wir betrachten die 1. Randwertaufgabe

$$(1.12) \qquad L u = - a_{jk} \cdot u_{jk} - b_j \cdot u_j + c \cdot u = r(x_i) \text{ in } B, \qquad u = f \text{ auf } \Gamma.$$

Die Funktion $\varphi(x_i)$ hat dann die Bedingungen

$$(1.13) \qquad L^* \varphi = L\varphi \geqq 0 \text{ in } B \quad \text{und} \quad \varphi > 0 \text{ auf } \Gamma$$

zu erfüllen.

Satz 2. *Sei $v(x_i) \in K$ eine die Differentialgleichung* (1.12) *exakt erfüllende Näherungslösung für $u(x_i)$ und nimmt $(u - v)$ auf Γ den Wert Null an, so gelten in $\overline{B}$ die Schranken*

$$(1.14) \qquad \varphi(x_i) \cdot \underset{\Gamma}{\mathrm{Min}} \frac{u-v}{\varphi} \leqq u(x_i) - v(x_i) \leqq \varphi(x_i) \cdot \underset{\Gamma}{\mathrm{Max}} \frac{u-v}{\varphi}$$

mit einer Funktion φ, die den Bedingungen (1.13) *genügt. Ist $L\varphi \equiv 0$ in B, so gelten die Ungleichungen* (1.14) *unabhängig davon, ob $(u - v)$ auf Γ den Wert Null annimmt oder nicht.*

Zum Beweise definieren wir wieder den Operator $\widetilde{L}z$ für $z = \frac{u-v}{\varphi}$ wie in (1.4) und erhalten mit $Tu - Tv = L(u - v) = 0$ und $G(x_i, z) = 0$ nach (1.5) $\widetilde{L}z = 0$. Auf diesen Operator (der linear ist mit dem Koeffizienten $L\varphi$ von z) können wir wegen $L\varphi \geqq 0$ in B die bekannte Fehlerabschätzung für die 1. Randwertaufgabe anwenden (s. COLLATZ [*1*], S. 375), d.h. es gilt $\underset{\Gamma}{\mathrm{Min}}\, z \leqq z \leqq \underset{\Gamma}{\mathrm{Max}}\, z$ in B, woraus sofort (1.14) folgt.

Zu beachten ist, daß wir bei der Anwendung des Satzes 2 die Funktion φ numerisch verwenden, während beim Satz 1 die Existenz einer solchen Funktion ausreichte. Man wird daher bei der Aufstellung von Fehlerschranken möglichst sorgfältig bei der Wahl der Funktion φ vorgehen.

Beispiel 3. Lineare partielle Differentialgleichung in zwei Veränderlichen $x_1 = x$, $x_2 = y$:

$$(1.15) \qquad \begin{aligned} &L u = -\Delta u - u = 1 \quad \text{für} \quad |x| < 1,\ |y| < 1 \\ &\text{und} \quad u = 0 \quad \text{für} \quad |x| = 1 \quad \text{und} \quad |y| = 1. \end{aligned}$$

Wir wählen $\varphi(x, y) = \cos \frac{x}{\sqrt{2}} \cdot \cos \frac{y}{\sqrt{2}} > 0$ in $\overline{B}$ mit $L\varphi = 0$ (es lassen sich leicht weitere Funktionen φ angeben, doch die hier gewählte dürfte die günstigste sein). Spezielle Lösungen der Differentialgleichung (1.15) lassen sich leicht bestimmen. Wir setzen die Näherungslösung $v(x, y)$ in der Form

$$v(x, y) = -1 + A(\cos x + \cos y) + \frac{B}{\cosh\tau}\left(\cosh\tau x \cdot \cos\frac{\pi}{2} y + \cos\frac{\pi}{2} x \cdot \cosh\tau y\right)$$

mit $\tau = \sqrt{\tfrac{1}{4}\cdot\pi^2 - 1} = 1{,}2113$ an, wobei die Symmetrieeigenschaften der Aufgabe bereits ausgenutzt wurden. Die Konstanten A und B bestimmen wir durch Kollokation so, daß v auf Γ „möglichst gut Null“ wird. Die Funktion

$$w(x) = -\frac{v(x, 1)}{\varphi(x, 1)} = \frac{1}{\varphi(x, 1)}\left[1 - A(\cos 1 + \cos x) - B\cos\frac{\pi}{2} x\right]$$

ist dann für die Fehlerabschätzung ausschlaggebend. Da $\varphi(x, 1)$ den kleinsten Wert bei $x = 1$ annimmt, werden wir die Kollokationspunkte ebenfalls nahe an den Rand $x = 1$ heranlegen. Aus der Forderung $w(0{,}5) = w(0{,}9) = 0$ erhalten wir $A = 0{,}91814$, $B = -0{,}42682$ und für diese Werte $w_{\max} < 0{,}01660$, $w_{\min} > -0{,}00784$, also in $|x| \leqq 1$, $|y| \leqq 1$ die Fehlerabschätzung

$$-0{,}0079 \cdot \varphi(x, y) \leqq u(x, y) - v(x, y) \leqq 0{,}0166 \cdot \varphi(x, y)$$

mit

$$v(x, y) = -1 + 0{,}91814(\cos x + \cos y) - 0{,}23351\left(\cosh\tau x \cdot \cos\frac{\pi}{2} y + \cos\frac{\pi}{2} x \cdot \cosh\tau y\right).$$

Speziell hat man für $u(0, 0)$ mit $v(0, 0) = 0{,}36925$ und $\varphi(0, 0) = 1$ die Schranken $|u(0, 0) - 0{,}3736| \leqq 0{,}0122$, d.h. der Näherungswert 0,3736 ist auf etwa 3% genau. — Durch eine andere Wahl der Kollokationspunkte ließe sich dieses Ergebnis wahrscheinlich noch etwas verbessern. Bei dem hier gemachten Ansatz mit nur zwei freien Parametern A und B läßt sich natürlich der Satz 1 direkt anwenden, indem man A und B so bestimmt, daß einmal $v_1(x, 1) \leqq 0$, ein andermal $v_2(x, 1) \geqq 0$ wird. Wir würden dann $A_1 = 0{,}88098$, $B_1 = -0{,}35701$, $A_2 = 0{,}92541$, $B_2 = -0{,}42541$ erhalten und damit $|u(0, 0) - 0{,}3784| \leqq 0{,}0070$, also für den Näherungswert 0,3784 einen Fehler von etwa 2%. Die direkte Anwendung des Satzes 1 macht aber bei einem Ansatz mit mehr als 2 freien Parametern größere Schwierigkeiten, so daß wir dann doch bequemer (1.14) heranziehen, dieses besonders, wenn z.B. eine elektronische Rechenanlage zur Verfügung steht, mit der das durch Kollokation erhaltene lineare Gleichungssystem gelöst und die minimalen und maximalen Werte von $\frac{u-v}{\varphi}$ auf dem Rande ermittelt werden können.

§ 2. Anfangswertaufgaben bei gewöhnlichen Differentialgleichungen

Wir betrachten den Operator

$$(2.1)\qquad Tu = u'' + F(x, u, u') \quad \text{in} \quad x_0 < x \leqq X \qquad \left(' = \frac{d}{dx}\right)$$

mit der in einem Bereich S des (x, u, u')-Raumes stetigen Funktion F. $u(x) \in K$ bedeutet: $u(x)$ ist in $x_0 \leqq x \leqq X$ mindestens einmal und in $x_0 < x \leqq X$ mindestens zweimal stetig differenzierbar.

Satz 3. *Für eine festgewählte Funktion* $v(x)\in K$ *existiere eine in* $x_0\leqq x\leqq X$ *stetig differenzierbare Funktion* $\alpha(x)$, *so daß*

$$G(x,z)=F(x,v+z,v'-\alpha z)-F(x,v,v')-(\alpha'-\alpha^2)\,z\geqq 0 \tag{2.2}$$

wird für $z\leqq 0$ *und* $x_0<x\leqq X$. *Gilt* $Tu<Tv$ *in* $x_0<x\leqq X$, $u<v$ *und* $u'+\alpha u<v'+\alpha v$ *für* $x=x_0$, *so gelten diese Ungleichungen in* $x_0\leqq x\leqq X$.

Beweis. Für die Funktionen $z(x)=u(x)-v(x)$ und $Z(x)=z'(x)+\alpha(x)\cdot z(x)$ wird $z(x_0)<0$ und $Z(x_0)<0$. Angenommen, es gilt nicht $z<0$ in $x_0\leqq x\leqq X$, so existiert ein Wert x_1 mit $x_0<x_1\leqq X$, so daß $z(x_1)=0$ und $z(x)<0$ für $x_0\leqq x<x_1$, woraus $Z(x_1)=z'(x_1)\geqq 0$ folgen würde. Wegen $Z(x_0)<0$ und der Stetigkeit von $Z(x)$ existiert dann ein Wert x_2 mit $x_0<x_2\leqq x_1$, $Z(x_2)=0$ und $Z(x)<0$ für $x_0\leqq x<x_2$, d.h. $Z'(x_2)\geqq 0$. Mit $Z'=(u-v)''+\alpha z'+\alpha' z$, $z'=Z-\alpha z$, $u=v+z$ und $u'=v'+Z-\alpha z$ wird

$$Z'(x)=Tu-Tv-F(x,v+z,v'+Z-\alpha z)+F(x,v,v')+\alpha Z+(\alpha'-\alpha^2)\,z.$$

Hieraus folgt für $x=x_2$ wegen $Tu-Tv<0$ und $z(x)\leqq 0$ in $x_0\leqq x\leqq x_2\leqq x_1$, $Z(x_2)=0$ und mit der durch (2.2) eingeführten Funktion: $Z'(x_2)<-G(x,z)\big|_{x=x_2}\leqq 0$. Dieses steht im Widerspruch zu $Z'(x_2)\geqq 0$, es existiert somit kein Wert x_2, für den $Z=0$ wird. Aus $Z(x)<0$ in $x_0\leqq x\leqq x_1$ ergibt sich dann $Z(x_1)=z'(x_1)<0$, d.h. auch $z(x)$ kann in $x_0\leqq x\leqq X$ den Wert Null nicht annehmen.

Zusatz. *Ist* (2.2) *für alle Funktionen* u *aus einem abgeschlossenen Bereich* S *des* (x,u,u')*-Raumes erfüllt, und ist* $F(x,u,u')$ *in* S *stetig differenzierbar, so folgt aus* $Tu\leqq Tv$ *in* $x_0<x\leqq X$, $u\leqq v$ *und* $u'+\alpha u\leqq v'+\alpha v$ *für* $x=x_0$ *die Gültigkeit dieser Ungleichungen in* $x_0\leqq x\leqq X$.

Der Satz 3 ist also auch gültig bei Einschluß der Gleichheitszeichen in den Voraussetzungen. Zum Beweise setzen wir $w=v+\varepsilon\cdot e^{\lambda(x-x_0)}$ mit $\varepsilon>0$. Für $x=x_0$ wird dann: $u\leqq v=w-\varepsilon<w$, $u'+\alpha u\leqq v'+\alpha v=w'+\alpha w-\varepsilon(\lambda+\alpha)<w'+\alpha w$ für $\lambda>-\alpha(x_0)$. Für Tv erhalten wir nach einer kurzen Rechnung $Tu\leqq Tv=Tw-\varepsilon\, e^{\lambda(x-x_0)}(\lambda^2+\widetilde{F}_u+\lambda\,\widetilde{F}_{u'})$, wobei die partiellen Ableitungen von F an einer Zwischenstelle zu nehmen sind. Wählen wir $\lambda>-\alpha(x_0)$ so groß, daß die eckige Klammer positiv wird, so gilt $Tu<Tw$ und nach Satz 3 folgt dann $u<w$ und $u'+\alpha u<w'+\alpha w$ in $x_0\leqq x\leqq X$. Hieraus ergeben sich für $\varepsilon\to 0$ die Behauptungen des Zusatzes.

Im folgenden wollen wir voraussetzen, daß $F(x,u,u')$ in S die Voraussetzungen des Zusatzes erfüllen möge. — Wir betrachten die Anfangswertaufgabe

$$Tu=u''+F(x,u,u')=r(x)\quad\text{für}\quad x_0<x,\qquad u(x_0)=u_0\quad\text{und}\quad u'(x_0)=u_0'. \tag{2.3}$$

Bei der Konstruktion von Näherungslösungen $v(x)$ wird man im allgemeinen von Funktionen ausgehen, die die Anfangsbedingungen exakt erfüllen. Gelingt es, zwei Funktionen $v_j(x)$ $(j=1,2)$ so zu bestimmen, daß

$$Tv_1\leqq Tu\leqq Tv_2\quad\text{für}\quad x_0<x\leqq X,\qquad v_j(x_0)=u_0\quad\text{und}\quad v_j'(x_0)=u_0' \tag{2.4}$$

gilt, so erhalten wir bei Erfüllung der Bedingung (2.2) Fehlerabschätzungen für die Funktion $u(x)$ und ihre Ableitung:

$$v_1\leqq u\leqq v_2\quad\text{und für}\quad\begin{cases}\alpha\geqq 0 & v_1'-\alpha(v_2-v_1)\leqq u'\leqq v_2'+\alpha(v_2-v_1)\\ \alpha\leqq 0 & v_1'\leqq u'\leqq v_2'\end{cases} \tag{2.5}$$

im Intervall $x_0 \leqq x \leqq X$. Die Schranken für die Ableitung u' werden also am günstigsten sein, wenn wir $\alpha(x) \leqq 0$ wählen können. Es soll jetzt untersucht werden, wann dieses möglich ist. Nach (2.2) erhalten wir (die partiellen Ableitungen sind hier an einer Zwischenstelle zu nehmen):

$$G(x, z) = (F_u - F_{u'}\,\alpha + \alpha^2 - \alpha')\,z \geqq 0 \quad \text{für} \quad z \leqq 0,$$

wenn in S die Bedingung

$$F_u - F_{u'} \cdot \alpha + \alpha^2 - \alpha' \leqq 0 \tag{2.6}$$

erfüllt ist.

1. Fall. $F_u \leqq 0$ in S, dann kann $\alpha = 0$ gewählt werden, die Anwendung des Satzes 3 ist hier besonders einfach. Allerdings dürfte dieser Fall bei Anwendungen aus Physik und Technik (z.B. nichtlineare Schwingungen oder der auf Zug (bzw. Druck) und Biegung einseitig eingespannte Balken) keine allzu große Bedeutung besitzen.

2. Fall. $F_u \leqq \omega^2$ in S mit $\omega^2 > 0$. Hier kann α nicht mehr gleich Null gesetzt werden. Wegen (2.5) werden wir möglichst $\alpha(x) \leqq 0$ wählen. Wir wollen uns bei der Diskussion auf den Sonderfall $F_{u'} = 0$ beschränken. Dann ist (2.6) erfüllt, wenn α der Bedingung

$$\omega^2 + \alpha^2 - \alpha' = 0 \quad \text{genügt, also} \quad \alpha(x) = \omega \cdot \tan\left[\omega(x - x_0) - \frac{\pi}{2} + C_1\right] \tag{2.7}$$

mit einer Integrationskonstanten $C_1 > 0$. Diese Funktion ist nicht mehr durchweg stetig für $x \geqq x_0$, d.h. wir erhalten zunächst nur Aussagen in einem gewissen Intervall $x_0 \leqq x \leqq x_1 \leqq x_0 + \frac{1}{\omega}(\pi - C_1)$ mit $0 < C_1 < \pi$ oder, wenn wir uns auf ein Intervall beschränken, in dem $\alpha(x) \leqq 0$ bleibt, $x_1 \leqq x_0 + \frac{1}{\omega}\left(\frac{\pi}{2} - C_1\right)$ mit $0 < C_1 < \frac{\pi}{2}$. In diesem letzten Fall erhalten wir die Schranken $v_1 \leqq u \leqq v_2$, $v_1' \leqq u' \leqq v_2'$ in $x_0 \leqq x \leqq x_1$. Jetzt könnte man ein nächstes Intervall $x_1 \leqq x \leqq x_2 \leqq x_1 + \frac{1}{\omega}\left(\frac{\pi}{2} - C_1\right)$ betrachten, in dem $\alpha_1(x) = \omega \cdot \tan\left[\omega(x - x_1) - \frac{\pi}{2} + C_1\right]$ gewählt wird und bestimmen die Funktionen $v_{j+2}(x)$ aus

$$\begin{gathered} T v_3 \leqq T u \leqq T v_4 \quad \text{in} \quad x_1 \leqq x \leqq x_2, \\ v_{j+2}(x_1) = v_j(x_1) \quad \text{und} \quad v_{j+2}' = v_j' + (-1)^j \alpha_1' \cdot (v_1 - v_2) \end{gathered} \tag{2.8}$$

für $x = x_1$, die dann die Schranken $v_3 \leqq u \leqq v_4$ und $v_3' \leqq u' \leqq v_4'$ in $x_1 \leqq x \leqq x_2$ liefern. In dieser Weise könnte man fortfahren und erhielte dann Schranken in einem Intervall $x_0 \leqq x \leqq X$. Praktisch jedoch dürfte diese Methode den meisten der anderen Methoden zur Fehlerabschätzung (z.B. Iterationsverfahren bei Anfangswertaufgaben) unterlegen sein.

Eine physikalisch anschauliche Deutung erhält (2.6) im Linearfall $F = 2bu' + cu$ mit b und c konstant. Dann setzen wir α ebenfalls konstant und erhalten

$$\alpha^2 - 2b\alpha + c \leqq 0. \tag{2.9}$$

Die linke Seite dieser Ungleichung ist gerade die Frequenzgleichung, die durch den Ansatz $u = e^{-\alpha x}$ erhalten wird. (2.9) läßt sich für $c > 0$ nicht mehr durch reelle α-Werte erfüllen, falls $b^2 - c^2 < 0$ ist. Dieses ist der bekannte Schwingungsfall. Es wäre möglich, daß sich aus (2.6) entsprechend ein Kriterium für das Eintreten eines Schwingungsvorganges gewinnen läßt, doch hierüber sind — ausgehend von diesen Entwicklungen in §2 — noch keine weiteren Untersuchungen angestellt worden.

Beispiel 4. Lineare Differentialgleichung

$$(2.10)\qquad T u = (2 - x^2)\, u'' + u = 3 - 2x \quad \text{für} \quad 0 < x, \qquad u(0) = u'(0) = 0.$$

$T u$ ist hier anders definiert als in (2.1), wo der Koeffizient bei u'' gleich 1 gesetzt wurde. Da aber $2 - x^2 > 0$ in $0 \leqq x < \sqrt{2}$ ist, gilt der Satz 3 entsprechend in diesem Intervall für den hier definierten Operator. Wir wollen uns im folgenden auf das Intervall $0 \leqq x \leqq 1$ beschränken. Mit $F_u = \frac{1}{2 - x^2} \leqq 1$ wählen wir $\alpha(x) = \tan(x - 1)$. Dann ist die Bedingung (2.6) erfüllt, und wir können die in (2.5) angegebenen Abschätzungen für $\alpha \leqq 0$ benutzen. Der Ansatz $v(x) = a x^2 + b x^3 + c x^4 + d x^5 + e x^6$ erfüllt die Anfangsbedingungen und liefert

$$T v - T u = (4a - 3) + (12b + 2)\, x + (24c - a)\, x^2 + (40d - 5b)\, x^3 + \\ + (60e - 11c)\, x^4 - 19 d x^5 - 29 e\, x^6$$

oder mit $a = \frac{3}{4}$, $b = -\frac{1}{6}$, $c = \frac{1}{32}$, $d = -\frac{1}{48}$:

$$T v - T u = x^4 \left[\left(60e - \tfrac{11}{32}\right) + \tfrac{19}{48} x - 29 e\, x^2\right].$$

Für $e_1 = -\frac{5}{62 \cdot 48}$ (aus $T v - T u = 0$ für $x = 1$) folgt

$$T v_1 - T u = \frac{x^4}{48}\left(-\frac{1323}{62} + 19x + \frac{145}{62} x^2\right) \leqq 0$$

und für $e_2 = \frac{11}{48 \cdot 40}$ $\left(\text{bestimmt aus } \frac{T v - T u}{x^4} = 0 \text{ für } x \to 0\right)$:

$$T v_2 - T u = \frac{x^5}{48}\left(19 - \frac{319}{40} x\right) > 0.$$

Insgesamt erhalten wir dann mit

$$v(x) = \frac{1}{2}(v_1 + v_2) = \frac{3}{4} x^2 - \frac{1}{6} x^3 + \frac{1}{32} x^4 - \frac{1}{48} x^5 + \frac{241}{192 \cdot 620} x^6$$

in $0 \leqq x \leqq 1$ die Fehlerabschätzung

$$|u(x) - v(x)| \leqq \frac{441}{192 \cdot 620} x^6 \quad \text{und} \quad |u'(x) - v'(x)| \leqq \frac{441}{64 \cdot 310} x^5.$$

Speziell für $x = 1$: $|u(1) - 0{,}6958| \leqq 0{,}0037$, $|u'(1) - 1{,}0330| \leqq 0{,}0223$.

Anmerkung. Über Monotoniesätze bei Anfangswertaufgaben hyperbolischer und über Fehlerabschätzungen aus den Defekten bei Randwertaufgaben elliptischer Differentialgleichungen soll in einer späteren Arbeit berichtet werden.

Literatur

[1] Collatz, L.: Numerische Behandlung von Differentialgleichungen, 2. Aufl. Berlin-Göttingen-Heidelberg: Springer 1955.

[2] Collatz, L.: Fehlerabschätzungen bei der 1. Randwertaufgabe bei elliptischen Differentialgleichungen. Z. angew. Math. Mech. **32**, 202—211 (1952).

[3] Meyer, A.: Fehlerabschätzungen bei Randwertaufgaben mit elliptischer Differentialgleichung. Diss. Univ. Hamburg 1959.

[4] Schröder, J.: Fehlerabschätzungen bei gewöhnlichen und partiellen Differentialgleichungen. Arch. Rational Mech. Anal. **2**, 367—392 (1958/59).

[5] Schröder, J.: Vom Defekt ausgehende Fehlerabschätzungen bei Differentialgleichungen. Arch. Rational Mech. Anal. **3**, 219—228 (1959).

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Stability and Boundedness of Systems

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Introduction

Most problems in the theory of stability which have been discussed up to the present concern the behavior of the solutions relative to a fixed solution (*cf.* [2], [3], [5]), and most problems in the theory of boundedness concern the norms of solutions (*cf.* [6]). In this paper we discuss the same concepts relative to an arbitrary solution of the system and their relationships with the Liapunov function.

Now we consider a system of differential equations

$$\frac{dx}{dt} = F(t, x). \tag{1}$$

In (1) x denotes an n-dimensional vector, and $F(t, x)$ is a given vector field which is defined and continuous in the product space $I \times E_x^n$, where I is the interval $0 \leqq t < \infty$ and E_x^n is Euclidean n-space. Let $x = x(t; x_0, t_0)$ be a solution of (1) through the initial point (t_0, x_0). We represent the Euclidean norm of x by $\|x\|$.

I. Definitions

In this section we give the following definitions:

(*i*) If corresponding to each $\alpha > 0$ and each $t_0 \geqq 0$ there exists a positive function $\beta(t_0, \alpha)$ which for each $\alpha > 0$ is continuous in t_0 and is such that

$$\|x(t; x_0, t_0) - x(t; \bar{x}_0, t_0)\| \leqq \beta(t_0, \alpha)$$

for all $\|x_0 - \bar{x}_0\| \leqq \alpha$ and $t \geqq t_0$, the system (1) is said to be *equi-distance-bounded*.

(*ii*) When $\beta = \beta(\alpha)$ in (*i*) is independent of t_0, the system (1) is said to be *uniform-distance-bounded*.

(*iii*) If corresponding to each $\alpha > 0$ and each $t_0 \geqq 0$ there exist positive numbers B and $T(t_0, \alpha)$ such that

$$\|x(t; x_0, t_0) - x(t; \bar{x}_0, t_0)\| < B$$

for all $\|x_0 - \bar{x}_0\| \leqq \alpha$ and $t > t_0 + T(t_0, \alpha)$, the system (1) is said to be *equiultimately distance-bounded for the bound B*.

(*iv*) When $T = T(\alpha)$ in (*iii*) is independent of t_0, the system (1) is said to be *uniform-ultimately distance-bounded for the bound B*.

(*v*) If corresponding to each $\varepsilon>0$ and each $t_0\geqq 0$ there exists a positive number $\delta(t_0, \varepsilon)$ such that

$$\|x(t; x_0, t_0)-x(t; \bar{x}_0, t_0)\|<\varepsilon$$

for all $\|x_0-\bar{x}_0\|<\delta$ and $t\geqq t_0$, the system (1) is said to be *equi-stable*, where δ is continuous in t_0.

(*vi*) When $\delta=\delta(\varepsilon)$ in (*v*) is independent of t_0, the system (1) is said to be *uniform-stable*.

(*vii*) If corresponding to each $\varepsilon>0$ and each $t_0\geq 0$ there exist positive numbers $\delta_0(t_0)$ and $T(t_0, \varepsilon)$ such that

$$\|x(t; x_0, t_0)-x(t; \bar{x}_0, t_0)\|<\varepsilon$$

for all $\|x_0-\bar{x}_0\|<\delta_0(t_0)$ and $t>t_0+T(t_0, \varepsilon)$, the system (1) is said to be *quasi-equiasymptotically stable*.

(*viii*) When every solution is stable in the sense of LIAPUNOV and the system (1) is equi-distance-bounded and moreover the system is quasi-equiasymptotically stable, the system (1) is said to be *equiasymptotically stable*.

(*ix*) When δ_0 and $T=T(\varepsilon)$ in (*vii*) are independent of t_0, the system (1) is said to be *quasi-uniform-asymptotically stable*.

(*x*) When the system (1) is uniform-stable and quasi-uniform-asymptotically stable, the system (1) is said to be *uniform-asymptotically stable*.

To simplify the statements, we shall use "*D-bounded*" in place of "*distance-bounded*".

We can also define the quasi-equiasymptotic stability of the system and the quasi-uniform-asymptotic stability of the system as follows:

(*xi*) If for any $\varepsilon>0$, any $\alpha>0$ and for any solution $x=x(t; x_0, t_0)$ of (1) there exists a positive number $T(t_0, \varepsilon, \alpha)$ such that if $\|x_0-\bar{x}_0\|\leqq\alpha$ and $t>t_0+T(t_0, \varepsilon, \alpha)$, we have

$$\|x(t; x_0, t_0)-x(t; \bar{x}_0, t_0)\|<\varepsilon,$$

the system (1) is said to be *quasi-equiasymptotically stable*.

(*xii*) When T in (*xi*) depends only on ε and α but is independent of t_0, the system (1) is said to be *quasi-uniform-asymptotically stable*.

Now we show that (*xi*) and (*xii*) are equivalent to (*vii*) and (*ix*), respectively.

Lemma 1. *Definition* (*vii*) *is equivalent to Definition* (*xi*).

Proof. It is clear that (*xi*) implies (*vii*), because it is sufficient that we consider some α as δ_0. We shall show that (*vii*) implies (*xi*). For any $\alpha>0$, there exists a smallest integer N such that $\alpha/\delta_0(t_0)<N$. For any $\varepsilon>0$, we can find a positive number $T(t_0, \varepsilon/N)$ such that $\|x_0-\bar{x}_0\|<\delta_0(t_0)$ and $t>t_0+T(t_0, \varepsilon/N)$ imply $\|x(t; x_0, t_0)-x(t; \bar{x}_0, t_0)\|<\varepsilon/N$. Then if $\|x_0-\bar{x}_0\|\leqq\alpha$, by subdividing the line segment joining x_0 and $\bar{x}_0$ we infer that $\|x(t; x_0, t_0)-x(t; \bar{x}_0, t_0)\|<\varepsilon$ for $t>t_0+T(t_0, \varepsilon/N)$.

Lemma 2. *Definition* (*ix*) *is equivalent to Definition* (*xii*).

Since the proof is the same as that of Lemma 1, it is omitted.

Remark 1. The above definitions of stability and D-boundedness imply that all the solutions of (1) are continuable to the right; that is to say, all the solutions are defined for $t_0 \leqq t < \infty$. Conversely, when a solution is not continuable to the right, then the conditions in the definition are not fulfilled (*cf.* pp. 136—137 in [*6*]).

Remark 2. If we suppose that $F(t, x)$ is continuous when $0 \leqq t < a$, $\|x\| < \infty$ $(a \leqq \infty)$, we can discuss the solutions when $0 \leqq t < a$.

II. Relations

In this section we shall discuss some relations between stability and D-boundedness. Of course, it is clear that the D-boundedness of a system does not imply the stability of the system (Ex. 8, p. 113 in [*6*]).

Lemma 3. *If the system* (1) *is equi-stable, it is equi-D-bounded.*

Proof. By the equi-stability of the system, for some $\varepsilon > 0$, there exists a positive number $\delta(t_0)$ such that if $\|x_0 - \bar{x}_0\| < \delta(t_0)$, we have for $t \geqq t_0$

$$\|x(t; x_0, t_0) - x(t; \bar{x}_0, t_0)\| < \varepsilon .$$

For any $\alpha > 0$, we can find the least integer N such that $\frac{\alpha}{\delta(t_0)} < N$. Then it is clear that if $\|x_0 - \bar{x}_0\| \leqq \alpha$, $\|x(t; x_0, t_0) - x(t; \bar{x}_0, t_0)\| \leqq N\varepsilon$, where $N\varepsilon$ is determined depending on α and t_0. Therefore, the system is equi-D-bounded.

Lemma 4. *If the system* (1) *is uniform-stable, it is uniform-D-bounded.*

Proof. In the proof of Lemma 3, for any $\alpha > 0$, N is determined independent of t_0, because we can choose δ independent of t_0.

To see the relations, we give an example.

Example 1. We consider a differential equation

$$\frac{dx}{dt} = \begin{cases} x^2 & (t = 0,\ x > 0), \\ \dfrac{1 + 2x\sin t - \sqrt{1 + 4x\sin t}}{2\sin^2 t}\cos t & (0 < t < \pi,\ x > 0), \\ -x^2 & (\pi \leqq t,\ x > 0), \\ -x & (0 \leqq t < \infty,\ x \leqq 0). \end{cases} \tag{2}$$

The general solution of (2) is $x = C^2 \sin t + C\ (C \geqq 0)$ in the region $0 \leqq t \leqq \pi$, $x \geqq 0$ and is $x = \frac{C}{1 + C(t - \pi)}$ $(C \geqq 0)$ in the region $\pi \leqq t < \infty$, $x \geqq 0$. Each solution of (2) is stable in the sense of LIAPUNOV, but the equation (2) is not equi-stable. Moreover the equation (2) is quasi-equiasymptotically stable, but it is not equi-D-bounded.

From this, we can see that

a) the stability of every solution in the sense of LIAPUNOV does not necessarily imply (*v*),

b) (*vii*) does not necessarily imply (*i*), and

c) (*iii*) does not necessarily imply (*i*).

The following example shows that the quasi-equiasymptotic stability of the system and the equi-D-boundedness of the system do not necessarily imply the equiasymptotic stability of the system.

Example 2. Now we consider a differential equation

$$(3)\ \frac{dx}{dt}=\begin{cases} 2\sqrt{x} & (0\leqq t\leqq 1,\ 0<x\leqq t^2),\\ 2t & (0\leqq t\leqq 1,\ t^2<x),\\ (2-t)\{t^2-4t+3+\sqrt{(t^2-4t+3)^2+4x}\} & (1<t<2,\ 0<x\leqq -t^2+4t-2),\\ -2t+4 & (1<t<2,\ -t^2+4t-2<x),\\ -(t-2)\,x & (2\leqq t<\infty,\ 0<x),\\ -x & (0\leqq t<\infty,\ x\leqq 0). \end{cases}$$

In the region $1\leqq t\leqq 2$, $0\leqq x\leqq -t^2+4t-2$, the solutions of this equation are

$$x=-\beta t^2+4\beta t+(\beta^2-3\beta) \qquad (\beta\geqq 0).$$

The equation (3) is quasi-equiasymptotically stable and is equi-D-bounded, but the solution $x\equiv 0$ is not stable in the sense of LIAPUNOV.

However, in the case where the system (1) is quasi-equiasymptotically stable, for any solution $x=x(t;x_0,t_0)$ and any $\varepsilon>0$, if $\|x_0-\bar{x}_0\|\leqq\alpha$ ($\alpha>0$: constant) and $t>t_0+T(t_0,\varepsilon)$, we have $\|x(t;x_0,t_0)-x(t;\bar{x}_0,t_0)\|<\varepsilon$. Then if every solution of the Cauchy-problem for (1) is unique in the interval $t_0\leqq t\leqq t_0+T(t_0,\varepsilon)$, there exists a positive number $\delta(<\alpha)$ such that if $\|x_0-\bar{x}_0\|<\delta$, we have $\|x(t;x_0,t_0)-x(t;\bar{x}_0,t_0)\|<\varepsilon$. Therefore if $\|x_0-\bar{x}_0\|<\delta$, we have $\|x(t;x_0,t_0)-x(t;\bar{x}_0,t_0)\|<\varepsilon$ for $t\geqq t_0$; that is to say, every solution of (1) is stable in the sense of LIAPUNOV.

Lemma 5. *If every solution of the Cauchy-problem for* (1) *is unique, the quasi-equiasymptotic stability of the system implies the stability of every solution in the sense of* LIAPUNOV. *Therefore, in this case, if the system* (1) *is equi-D-bounded and is quasi-equiasymptotically stable, it is equiasymptotically stable.*

Example 2 shows the situation also for equiasymptotic stability in the large [*6*].

Example 1 shows also that

a) (*ix*) does not imply (*x*) (T in Example 1 is independent of t_0),

b) (*ix*) does not imply (*i*) and

c) (*iv*) does not imply (*i*).

The following proposition is easily verified.

Proposition 1. (a) (*viii*) implies the equiasymptotic stability in the large of every solution.

(b) (*x*) implies the uniform-asymptotic stability in the large of every solution.

From Example 1, it is clear that the converses are not true.

As was stated before, the equiultimate D-boundedness of the system does not in general imply the equi-D-boundedness of the system. However, we have

Proposition 2. When the system (1) is linear, or, more generally, when $F(t, x)$ satisfies a Lipschitz condition

$$\|F(t, x) - F(t, \bar{x})\| \leqq L(t) \|x - \bar{x}\|,$$

where $L(t)$ is continuous and $x \in E^n_x$, $\bar{x} \in E^n_x$, then *(iii)* implies *(i)*.

Proof. By *(iii)*, for any $\alpha > 0$ and any solution $x = x(t; x_0, t_0)$, there exists a $T(t_0, \alpha)$ such that if $\|x_0 - \bar{x}_0\| \leqq \alpha$ and $t > t_0 + T(t_0, \alpha)$, we have $\|x(t; x_0, t_0) - x(t; \bar{x}_0, t_0)\| < B$, where B is a fixed constant. In the interval $t_0 \leqq t \leqq t_0 + T(t_0, \alpha)$, by the Lipschitz condition,

$$\begin{aligned}\|x(t; x_0, t_0) - x(t; \bar{x}_0, t_0)\| &\leqq \|x_0 - \bar{x}_0\| \exp\left(\int_{t_0}^{t_0 + T(t_0, \alpha)} L(s)\, ds\right) \\ &\leqq \alpha \exp\left(\int_{t_0}^{t_0 + T(t_0, \alpha)} L(s)\, ds\right).\end{aligned}$$

Therefore, if $t \geqq t_0$,

$$\|x(t; x_0, t_0) - x(t; \bar{x}_0, t_0)\| \leqq \max\left(B, \alpha \exp\left(\int_{t_0}^{t_0 + T(t_0, \alpha)} L(s)\, ds\right)\right).$$

This means that *(iii)* implies *(i)*.

In the case where (1) is a linear system

$$\frac{dx}{dt} = A(t)\, x + f(t), \tag{4}$$

by the transformation

$$x = y + x(t; x_0, t_0) \tag{5}$$

the equation (4) is transformed into the equation

$$\frac{dx}{dt} = A(t)\, y, \tag{6}$$

where $x = x(t; x_0, t_0)$ is a solution of (4). Therefore the behavior of solutions of (4) relative to the solution $x(t; x_0, t_0)$ is the same as that of solutions of (6) relative to the solution $y \equiv 0$. Moreover, for any solution of (4), we obtain the same equation (6). For example, if a solution $x = x(t; x_0, t_0)$ of (4) is stable in the sense of Liapunov, then corresponding to each $\varepsilon > 0$ there is a positive number δ such that if $\|x_0 - \bar{x}_0\| < \delta$, we have $\|x(t; x_0, t_0) - x(t; \bar{x}_0, t_0)\| < \varepsilon$ for all $t \geqq t_0$. Hence if $\|y_0\| < \delta$, we have $\|y(t; y_0, t_0)\| < \varepsilon$ for all $t \geqq t_0$. Therefore, the solution $y(t) \equiv 0$ of (6) is stable in the sense of Liapunov, and then, from the stability of $y \equiv 0$, we can also see that every solution of (4) is stable in the sense of Liapunov and that the system (4) is equi-stable, because we can suppose that $t_0 = 0$. Therefore, we have

Proposition 3. (a) The stability of $y \equiv 0$ of (6) in the sense of Liapunov, the stability of one solution of (4) in the sense of Liapunov and the equi-stability of the system (4) are equivalent.

(b) The uniform-stability of $y \equiv 0$ of (6), the uniform-stability of one solution of (4) and the uniform-stability of the system (4) are equivalent.

Corresponding to the *D*-boundedness of the system, when we discuss the behavior of solutions relative to some fixed solution, we shall say that the solution is *D-bounded*.

We can infer the following proposition in the same way as Proposition 3.

Proposition 4. (a) The boundedness of (6), the D-boundedness of one solution of (4) and the equi-D-boundedness of the system (4) are equivalent.

(b) The uniform-boundedness of (6), the uniform-D-boundedness of one solution of (4) and the uniform-D-boundedness of the system (4) are equivalent.

Since the stability of (6) and the uniform-stability of (6) are equivalent to the equi-boundedness of (6) and the uniform-boundedness of (6), respectively (*cf.* p. 103 in [*6*]), we have

Proposition 5. In a linear system, the stability in the sense of LIAPUNOV, the equi-stability of the system, the D-boundedness and the equi-D-boundedness of the system are equivalent. Moreover, the uniform-stability, the uniform-stability of the system, the uniform-D-boundedness and the uniform-D-boundedness of the system are equivalent.

Example 3. Now we consider a linear equation,

$$\frac{dx}{dt} = (6t\cos t - 2t)\,x. \tag{7}$$

The general solution is $x = x_0 \exp(-6t_0 \sin t_0 - 6\cos t_0 + t_0^2 + 6t\sin t + 6\cos t - t^2)$. If we put $-6t_0\sin t_0 - 6\cos t_0 + t_0^2 = c(t_0)$, we have $|x| \leqq |x_0|\, e^{15+c(t_0)}$. From this, we can see that the solution $x(t) \equiv 0$ is stable in the sense of LIAPUNOV. Therefore, the equation (7) is equi-D-bounded by Proposition 5. However, since $\left|x\left(2n\pi + \frac{\pi}{2};\, x_0,\, 2n\pi\right)\right| \to \infty$ with n, the solution $x(t) \equiv 0$ is not uniformly stable and hence the equation is not uniform-stable. This example shows that the equi-stability of the system and the uniform-stability of the system are different concepts. Of course, the equation (7) is equi-D-bounded, but it is not uniform-bounded.

In the same way as the above we can infer the following proposition.

Proposition 6. In a linear system, the asymptotic stability in the sense of LIAPUNOV, the equiasymptotic stability of the system, the ultimate D-boundedness and the equiultimate D-boundedness of the system are equivalent. Moreover, the quasi-uniform-asymptotic stability (*cf.* [*1*]), the quasi-uniform-asymptotic stability of the system, the uniform-ultimate D-boundedness and the uniform-ultimate D-boundedness of the system are equivalent. The uniform-asymptotic stability is equivalent to the uniform-asymptotic stability of the system.

Remark. In a linear system, the equiasymptotic stability of the system and the equiasymptotic stability in the large are equivalent, and also the uniform-asymptotic stability of the system and the uniform-asymptotic stability in the large are equivalent.

III. Relations with the Liapunov function

In this section we discuss the use of the Liapunov function $V(t, x, y)$ in the present context. For this purpose, we consider the Liapunov function $V(t, x, y)$ with respect to the system

$$\frac{dx}{dt} = F(t, x), \qquad \frac{dy}{dt} = F(t, y), \tag{8}$$

where the space considered is the product space $I \times E_x^n \times E_y^n$. Corresponding to the Liapunov function $V(t, x, y)$, we define the function

$$V'(t, x, y) = \overline{\lim_{h \to +0}} \frac{1}{h} \{V(t+h, x+hF(t, x), y+hF(t, y)) - V(t, x, y)\}.$$

Then we have theorems analogous to those for stability and boundedness [*4*], [*5*], [*6*].

Theorem 1. *We suppose that there exists a Liapunov function $V(t, x, y)$ defined in $0 \leqq t < \infty$, $\|x-y\| \leqq H$ $(H>0)$ in the product space $I \times E_x^n \times E_y^n$ and satisfying the following conditions;*

1° $b(\|x-y\|) \leqq V(t, x, y) \leqq a(t, \|x-y\|)$, *where $a(t, r)$ and $b(r)$ are positive for $r>0$ and are continuous increasing functions, and $a(t, r) \to 0$ as $r \to 0$,*

2° $V(t, x, y) \in C_0(x, y)$ (cf. p. 100 in [*6*]),

3° $V'(t, x, y) \leqq 0$.

Then the system (1) *is equi-stable.*

Proof. For any $\varepsilon > 0$, if $\|x-y\| = \varepsilon$, we have $b(\varepsilon) \leqq V(t, x, y)$. Since $a(t, r) \to 0$ as $r \to 0$, there exists a positive number $\delta(t_0, \varepsilon)$ such that if $r \leqq \delta(t_0, \varepsilon)$, then

$$a(t_0, r) < b(\varepsilon).$$

For some solution $x = x(t; x_0, t_0)$, we suppose that some solution $x = x(t; \bar{x}_0, t_0)$ such that $\|x_0 - \bar{x}_0\| < \delta(t_0, \varepsilon)$ satisfies $\|x(t_1; x_0, t_0) - x(t_1 : \bar{x}_0, t_0)\| = \varepsilon$ at some t, say t_1. Then using the conditions 2° and 3°, we obtain an inequality

$$b(\varepsilon) \leqq V(t_1, x(t_1; x_0, t_0),\ x(t_1; \bar{x}_0, t_0)) \leqq V(t_0, x_0, \bar{x}_0) \leqq a(t_0, \delta) < b(\varepsilon).$$

This statement is a contradiction. It follows that if $\|x_0 - \bar{x}_0\| < \delta(t_0, \varepsilon)$ and $t \geqq t_0$, we have

$$\|x(t; x_0, t_0) - x(t; \bar{x}_0, t_0)\| < \varepsilon.$$

Theorem 2. *If $a(t, r)$ in Theorem 1 is independent of t, the system* (1) *is uniform-stable.*

Theorem 3. *We suppose that there exists a Liapunov function $V(t, x, y)$ defined in the product space $I \times E_x^n \times E_y^n$ and satisfying the following conditions;*

1° $b(\|x-y\|) \leqq V(t, x, y) \leqq a(t, \|x-y\|)$, *where $a(t, r)$ and $b(r)$ are non-negative continuous functions and $b(r) \to \infty$ as $r \to \infty$,*

2° $V(t, x, y) \in C_0(x, y)$,

3° $V'(t, x, y) \leqq 0$.

Then the system (1) *is equi-D-bounded.*

Theorem 4. *If $V(t, x, y)$ in Theorem 3 is defined when $0 \leqq t < \infty$, $\|x-y\| \geqq R_0$ ($R_0 > 0$ may be sufficiently large) in $I \times E_x^n \times E_y^n$, and if $a(t, r)$ is independent of t, the system* (1) *is uniform-D-bounded.*

Next we discuss the equiultimate D-boundedness of the system and the equiasymptotic stability of the system.

Theorem 5. *We assume that $F(t, x) \in C_0(x)$. In order that the system* (1) *be equi-D-bounded and equiultimately D-bounded, it is necessary and sufficient that* *there exist a positive number B and a non-negative Liapunov function $V(t, x, y)$ satisfying the following conditions in $I \times E_x^n \times E_y^n$;*

1° $b(\|x-y\|) \leqq V(t, x, y)$ for $\|x-y\| \geqq B$, where $b(r)$ is a continuous function which is positive and increasing when $r > B$, and where $b(r) \to \infty$ as $r \to \infty$,

2° $V(t, x, y) \leqq a(t, \|x-y\|)$, where $a(t, r)$ is a continuous increasing function,

3° $V(t, x, y) \in C_0(t, x, y)$ and $V'(t, x, y) \leqq -cV(t, x, y)$, where c is a positive constant.

Proof. The proof of sufficiency is the same as that of Theorem 11 in [6]. Now we shall show that the condition is necessary. Let Ω_η be the domain

$$0 \leqq t \leqq \eta, \quad \|x\| \leqq \eta, \quad \|y\| \leqq \eta.$$

If $(t_0, x_0, y_0) \in \Omega_\eta$, we have $\|x_0 - y_0\| \leqq 2\eta$. There exists a $T(\eta)$ such that if $t > T(\eta)$, we have $\|x(t; x_0, t_0) - y(t; y_0, t_0)\| < B$, where B is the bound for the equi-D-boundedness of the system: for a solution $z = z(t; z_0, t_0)$ of the system (8), where $z = (x, y)$, is continuable to $t = \eta$ and hence $\|z(\eta; z_0, t_0)\| \leqq \bar{\eta}$, whence we have $\|x\| \leqq \bar{\eta}$, $\|y\| \leqq \bar{\eta}$, and therefore it is sufficient that we take $\eta + T(2\bar{\eta})$, where $T(2\bar{\eta})$ is one for $t = \eta$, $\|x-y\| \leqq 2\bar{\eta}$. We can assume that $T(\eta)$ is a non-negative continuous function defined for $\eta \geqq 0$ and is monotone increasing in η.

For $(t_0, x_0, y_0) \in \Omega_\eta$ and $t_0 \leqq t \leqq T(\eta)$, there is $\beta(\eta)$ such that

$$\sup\{\|x(t; x_0, t_0)\|, \|y(t; y_0, t_0)\|\} \leqq \beta(\eta),$$

where $\beta(\eta)$ is a positive continuous increasing function. Moreover, if we put

$$\max_{\substack{0 \leqq t \leqq \eta \\ 0 \leqq \|x\| \leqq \beta(\eta)}} \|F(t, x)\| = F^*(\eta),$$

$F^*(\eta)$ is continuous and increasing, and we may assume that $F^*(\eta) \geqq 1$. Moreover, let D_η be the set of points (t, x) such that

$$0 \leqq t \leqq T(\eta), \quad 0 \leqq \|x\| \leqq \beta(\eta).$$

Since $F(t, x) \in C_0(x)$, there is a positive constant $L(\eta)$ such that

$$\|F(t, x) - F(t, x')\| \leqq L(\eta)\, \|x - x'\|,$$

where $(t, x) \in D_\eta$ and $(t, x') \in D_\eta$. We can also assume that $L(\eta)$ is a positive continuous function and that it is increasing in η.

Now we consider a function $G(\zeta)$ as follows:

$$G(\zeta) = \begin{cases} \zeta - B & (\zeta \geqq B) \\ 0 & (0 \leqq \zeta < B). \end{cases}$$

Moreover we put

(9) $$2F^*(\eta) \exp\{(L(\eta)+1)\, T(\eta)\} + G(2\beta(\eta)) \exp\{T(\eta) + \eta\} = M(\eta).$$

Now we define $V(t, x, y)$ as follows:

(10) $$V(t, x, y) = \sup_\tau \left[G(\|x(t+\tau; x, t) - y(t+\tau; y, t)\|)\, e^\tau;\ 0 \leqq \tau\right].$$

It is clear that

(11) $$G(\|x-y\|) \leqq V(t, x, y).$$

Since the system (1) is equi-D-bounded and equiultimately D-bounded, there exist two positive constants $\tilde{\beta}(t, \alpha)$ and $T(t, \alpha)$ such that if $\|x-y\| \leqq \alpha$, we have for $\tau \geqq 0$

$$\|x(t+\tau; x, t) - y(t+\tau; y, t)\| \leqq \tilde{\beta}(t, \alpha),$$

and for $\tau > T(t, \alpha)$

$$\|x(t+\tau; x, t) - y(t+\tau; y, t)\| < B.$$

Therefore

$$V(t, x, y) \leqq G(\tilde{\beta}(t, \|x-y\|))\, e^{T(t, \|x-y\|)}, \tag{12}$$

and we can find a continuous function $a(t, r)$ such that $V(t, x, y) \leqq a(t, \|x-y\|)$.

In the same way as in Theorem 11 in [6], we see that $V(t, x, y) \in C_0(t, x, y)$ For instance, supposing that $(t, x, y) \in \Omega_\eta$ and $(t, x', y') \in \Omega_\eta$ and that $V(t, x, y) = G(\|x(t+\tau; x, t) - y(t+\tau; y, t)\|)\, e^{\tau}$, we have

$$\begin{aligned} V(t, x, y) - V(t, x', y') &\leqq G(\|x(t+\tau; x, t) - y(t+\tau; y, t)\|)\, e^{\tau} - \\ &\quad - G(\|x(t+\tau; x', t) - y(t+\tau; y', t)\|)\, e^{\tau} \\ &\leqq e^{\tau}\{\|x(t+\tau; x, t) - x(t+\tau; x', t)\| + \\ &\quad + \|y(t+\tau; y, t) - y(t+\tau; y', t)\|\} \\ &\leqq e^{T(\eta)}\{\|x-x'\|\, e^{L(\eta)\, T(\eta)} + \|y-y'\|\, e^{L(\eta)\, T(\eta)}\} \\ &\leqq e^{(L(\eta)+1)\, T(\eta)}\{\|x-x'\| + \|y-y'\|\}. \end{aligned}$$

In a similar way we obtain

$$V(t, x, y) - V(t, x', y') \geqq -e^{(L(\eta)+1)\, T(\eta)}\{\|x-x'\| + \|y-y'\|\}.$$

Therefore, by using (9), we have

$$|V(t, x, y) - V(t, x', y')| \leqq M(\eta)\{\|x-x'\| + \|y-y'\|\}.$$

Finally, for two points (t, x, y) and (t', x', y') which belong to Ω_η, we have

$$|V(t, x, y) - V(t', x', y')| \leqq M(\eta)\,[|t-t'| + \|x-x'\| + \|y-y'\|]. \tag{13}$$

Moreover, it is easy to show that

$$V'(t, x, y) \leqq -V(t, x, y). \tag{14}$$

Thus the theorem is proved.

When the system (1) is uniform-D-bounded and uniform-ultimately D-bounded, there exists a $\tilde{\beta}(\alpha)$ such that if $\|x-y\| \leqq \alpha$, we have for $\tau \geqq 0$

$$\|x(t+\tau; x, t) - y(t+\tau; y, t)\| \leqq \tilde{\beta}(\alpha),$$

and in addition there exists a $T(\alpha)$ such that if $\|x-y\| \leqq \alpha$, we have

$$\|x(t+\tau; x, t) - y(t+\tau; y, t)\| < B$$

for $\tau > T(\alpha)$. Therefore

$$V(t, x, y) \leqq G(\tilde{\beta}(\|x-y\|))\, e^{T(\|x-y\|)};$$

that is, there exists a continuous function $a(r)$ such that $V(t, x, y) \leqq a(\|x-y\|)$. Thus we have the following theorem:

Theorem 6. *We assume that $F(t, x) \in C_0(x)$. In order that the system* (1) *be uniform-D-bounded and uniform-ultimately D-bounded, it is necessary and sufficient that there exist a positive Liapunov function $V(t, x, y)$ satisfying the following conditions in $0 \leqq t < \infty$, $\|x-y\| \geqq R_0$ (R_0 may be sufficiently large):*

1° $b(\|x-y\|) \leqq V(t, x, y) \leqq a(\|x-y\|)$, *where $a(r)$ and $b(r)$ are positive continuous increasing functions and $b(r) \to \infty$ as $r \to \infty$,*

2° $V(t, x, y) \in C_0(t, x, y)$,

3° $V'(t, x, y) \leqq -c(\|x-y\|)$, *where $c(r)$ is a positive continuous function.*

Remark 1. In Theorems 5 and 6, for the sufficiency of the conditions we need not require that $F(t, x) \in C_0(x)$.

Remark 2. When $F(t, x)$ is autonomous or periodic in t, we can obtain a $V(t, x, y)$ which is autonomous or periodic in t. This is clear by the definition of $V(t, x, y)$.

By using Theorem 5, we can obtain the condition for the equiasymptotic stability of the system.

Theorem 7. *We assume that $F(t, x) \in C_0(x)$. In order that the system* (1) *be equiasymptotically stable, it is necessary and sufficient that there exists a Liapunov function $V(t, x, y)$ satisfying the following conditions in $I \times E_x^n \times E_y^n$;*

1° $V(t, x, y) = 0$ *if* $\|x-y\| = 0$ *and* $V(t, x, y) > 0$ *if* $\|x-y\| \neq 0$,

2° $b(\|x-y\|) \leqq V(t, x, y) \leqq a(t, \|x-y\|)$, *where $a(t, r)$ and $b(r)$ are continuous and $b(r)$ is an increasing function such that $b(r) > 0$ for $r > 0$ and $b(r) \to \infty$ with r,*

3° $V(t, x, y) \in C_0(t, x, y)$ *and* $V'(t, x, y) \leqq -cV(t, x, y)$, *where c is a positive constant.*

Proof. For the sufficiency of the conditions, we need not require $F(t, x) \in C_0(x)$. If we assume that $F(t, x) \in C_0(x)$, every solution of the Cauchy problem for (1) is unique, and hence, by Lemma 5, every solution is stable in the sense of LIAPUNOV. However, we can verify the stability without this assumption. Since $V(t, x, y)$ is continuous and vanishes at $x = y$, for any $\varepsilon > 0$ and for fixed t_0, x_0 there is a positive constant δ such that if $\|x_0 - y_0\| < \delta$, we have $V(t_0, x_0, y_0) < b(\varepsilon)$. From this, by using the conditions 2° and 3°, we can see that if $\|x_0 - y_0\| < \delta$, $\|x(t; x_0, t_0) - y(t; y_0, t_0)\| < \varepsilon$ for $t \geqq t_0$. Therefore, $x = x(t; x_0, t_0)$ is stable in the sense of LIAPUNOV.

Since B in Theorem 5 is chosen arbitrarily, it is clear that the condition is sufficient.

Now we show that the condition is necessary. Since the system is equi-ultimately D-bounded for any positive number ε, we write $T(\eta)$ in Theorem 5 as $T(\varepsilon, \eta)$, and we assume that $T(\varepsilon, \eta) = T(1, \eta)$ for $\varepsilon > 1$. Then $M(\eta)$ in Theorem 5 depends also on ε, and so we write it as $M(\varepsilon, \eta)$; that is,

$$M(\varepsilon, \eta) = 2F^*(\varepsilon, \eta) \exp\{(L(\varepsilon, \eta) + 1)\, T(\varepsilon, \eta)\} + G(\varepsilon, 2\beta(\varepsilon, \eta)) \exp\{T(\varepsilon, \eta) + \eta\}.$$

$M(\varepsilon, \eta)$ is a positive continuous function defined for $\varepsilon > 0$, $\eta \geqq 0$. By MASSERA's lemma in [5] (*cf.* p. 133 in [6]), there exist two functions $g(\varepsilon)$, $h(\eta) \in C_\infty$ in $[0, +\infty)$ such that $h(\eta) > 0$, $g(\varepsilon) > 0$ when $\varepsilon > 0$, $g(0) = 0$ and that

$$g(\varepsilon)\, M(\varepsilon, \eta) \leqq h(\eta). \tag{15}$$

To simplify the statements, we denote $G(\varepsilon, \zeta)$, $\beta(\varepsilon, \eta)$, $T(\varepsilon, \eta)$ and $g(\varepsilon)$ for $\varepsilon = 1/n$ $(n = 1, 2, \ldots)$ as $G_n(\zeta)$, $\beta_n(\eta)$, $T_n(\eta)$ and g_n respectively.

Now we put

$$V_n(t, x, y) = g_n \sup_\tau \left[G_n\left(\|x(t+\tau; x, t) - y(t+\tau; y, t)\|\right) e^\tau;\ 0 \leqq \tau\right]. \tag{16}$$

It is clear that

$$g_n G_n\left(\|x - y\|\right) \leqq V_n(t, x, y)$$

and

$$V_n(t, x, y) = 0 \quad \text{if} \quad x = y.$$

If $(t, x, y) \in \Omega_\eta$,

$$V_n(t, x, y) \leqq g_n G_n\left(2\beta_n(\eta)\right) e^{T_n(\eta)} \leqq h(\eta) \quad (\text{by } (15)). \tag{17}$$

On the other hand, by the equi-D-boundedness of the system and the equi-ultimate D-boundedness of the system, there exist two functions $\tilde{\beta}(t, \|x - y\|)$ and $\tilde{T}_n(t, \|x - y\|)$ such that

$$V_n(t, x, y) \leqq g_n G_n\left(\tilde{\beta}(t, \|x - y\|)\right) e^{\tilde{T}_n(t, \|x-y\|)}. \tag{18}$$

If $(t, x, y) \in \Omega_\eta$ and $(t', x', y') \in \Omega_\eta$,

$$|V_n(t, x, y) - V_n(t', x', y')| \leqq h(\eta)\left[|t - t'| + \|x - x'\| + \|y - y'\|\right].$$

Moreover,

$$V_n'(t, x, y) \leqq -V_n(t, x, y).$$

We are now going to obtain the desired function. We put

$$V(t, x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} V_n(t, x, y). \tag{19}$$

Then $V(t, x, y)$ is defined and continuous in the product space $I \times E_x^n \times E_y^n$ and if $x = y$, $V(t, x, y) = 0$. In the same way as in Theorem 21 in [6], we can show that there exists a continuous function $b(r)$ such that $b(\|x - y\|) \leqq V(t, x, y)$, $b(r) > 0$ for $r > 0$ and $b(r) \to \infty$ with r. From (18) we have

$$\begin{aligned} V_n(t, x, y) &\leqq g_n G_n\left(\tilde{\beta}(t, \|x - y\|)\right) e^{\tilde{T}_n(t, \|x-y\|)} \\ &\leqq \tilde{\beta}(t, \|x - y\|)\, g_n\, e^{\tilde{T}_n(t, \|x-y\|)}. \end{aligned} \tag{20}$$

On the other hand, if we put $\max(t, \|x - y\|) = \eta$, we may consider that the following inequality holds:

$$\tilde{T}_n(t, \|x - y\|) \leqq T_n(\eta).$$

Hence

$$e^{\tilde{T}_n(t, \|x-y\|)} \leqq e^{T_n(\eta)} \leqq M_n(\eta).$$

Therefore

$$\begin{aligned} g_n e^{\tilde{T}_n(t, \|x-y\|)} &\leqq h(\eta) \\ &\leqq h\left(\eta(t, \|x - y\|)\right), \end{aligned}$$

whence by aid of (20) we can see that there exists a function $a(t, r)$ such that

$$V_n(t, x, y) \leqq a\left(t, \|x - y\|\right).$$

Hence

$$V(t, x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} V_n(t, x, y) \leqq a(t, \|x-y\|) \sum_{n=1}^{\infty} \frac{1}{2^n}$$
$$\leqq a(t, \|x-y\|).$$

As in Theorem 21 in [6], we can see that

$$V(t, x, y) \in C_0(t, x, y) \quad \text{and} \quad V'(t, x, y) \leqq -V(t, x, y).$$

Next we shall discuss the uniform-asymptotic stability of the system. In this case, the system (1) is uniform-stable, and so by Lemma 4 it is uniform-D-bounded. Therefore there exist two continuous functions $\alpha(r)$ and $\beta(r)$ such that for $\tau \geqq 0$

$$\alpha(\|x-y\|) \leqq \|x(t+\tau; x, t) - y(t+\tau; y, t)\| \leqq \beta(\|x-y\|)$$

and $\beta(r) \to 0$ as $r \to 0$. Since the system is uniform-asymptotically stable, T is determined as a function of $\|x-y\|$ only. Therefore, in place of (20) we have

$$V_n(t, x, y) \leqq g_n G_n(\beta(\|x-y\|)) e^{T_n(\|x-y\|)}.$$

Since T_n is independent of t, we may assume that for $0 \leqq t < \infty$

$$e^{T_n(\|x-y\|)} \leqq M_n(\|x-y\|).$$

Hence

$$V_n(t, x, y) \leqq G_n(\beta(\|x-y\|)) g_n e^{T_n(\|x-y\|)}$$
$$\leqq \beta(\|x-y\|) g_n M_n(\|x-y\|)$$
$$\leqq \beta(\|x-y\|) h(\|x-y\|).$$

Therefore we obtain the following theorem.

Theorem 8. *We assume that $F(t, x) \in C_0(x)$. In order that the system* (1) *be uniform-asymptotically stable, it is necessary and sufficient that there exists a Liapunov function satisfying the following conditions in $I \times E_x^n \times E_y^n$;*

1° *$V(t, x, y) = 0$ if $\|x-y\| = 0$ and $V(t, x, y) > 0$ if $\|x-y\| \neq 0$,*

2° *$b(\|x-y\|) \leqq V(t, x, y) \leqq a(\|x-y\|)$, where $a(r)$ and $b(r)$ are continuous in r and $b(r)$ is an increasing function such that $b(r) > 0$ for $r > 0$ and $b(r) \to \infty$ with r and that $a(r) \to 0$ as $r \to 0$,*

3° *$V(t, x, y) \in C_0(t, x, y)$ and $V'(t, x, y) \leqq -c(\|x-y\|)$, where $c(r)$ is continuous and positive definite.*

Remark 1. When $F(t, x)$ is autonomous or periodic in t, we can obtain a V which is autonomous or periodic in t.

Remark 2. By Definition (x), it is sufficient that $V(t, x, y)$ in Theorem 8 be defined when $0 \leqq t < \infty$, $\|x-y\| \leqq H$, where H is some positive constant.

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References

[1] ANTOSIEWICZ, H. A.: A survey of LYAPUNOV's second method. Annals of Math. Studies 41. pp. 141—166. Princeton Univ. Press 1958.

[2] CESARI, L.: Asymptotic Behavior and Stability Problems in Ordinary Differential Equations. Ergebnisse der Math. und ihrer Grenzgebiete, Heft 16, Springer 1959.

[3] LEFSCHETZ, S.: Differential Equations: Geometric Theory. Interscience 1957.

[4] LIAPOUNOFF, A. M.: Problème Général de la Stabilité du Mouvement. Annals of Math. Studies 17. Princeton Univ. Press 1949.

[5] MASSERA, J. L.: Contributions to stability theory. Ann. of Math. **64**, 182—206 (1956).

[6] YOSHIZAWA, T.: LIAPUNOV's function and boundedness of solutions. Funkcialaj Ekvacioj **2**, 95—142 (1959).

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Fehlerabschätzungen für die Ableitungen bei Randwertaufgaben mit linearer elliptischer Differentialgleichung

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Wie man Schranken für die Ableitungen der Lösung einer Randwertaufgabe aufstellen kann, wenn für die Lösung selbst bereits eine Abschätzung vorliegt, ist für die Potentialgleichung schon von O. D. KELLOGG ([*8*], S. 227), für allgemeinere elliptische lineare Differentialgleichungen von J. SCHAUDER [*10*], [*11*], für Systeme von elliptischen Differentialgleichungen von A. DOUGLIS und L. NIRENBERG [*4*] gezeigt worden.

Dem Gedankengang der Arbeit [*10*] von J. SCHAUDER folgend, werden in Nr. 1 Abschätzungsformeln mit zahlenmäßig bestimmten Konstanten angegeben, die im Inneren des betrachteten Gebietes gelten. In Spezialfällen lassen sich die Schranken auch auf den Rand ausdehnen.

Die mathematische Behandlung von Strömungen führt oft auf reine zweite Randwertaufgaben, bei denen die Konstante Lösung der homogenen Aufgabe ist. Von besonderem Interesse sind dann meist die Tangentialableitungen auf dem Rande. Für spezielle solche Aufgaben in der Ebene wird in Nr. 2 gezeigt, wie man Schranken für die Tangentialableitung gewinnen kann.

Nr. 3 bringt einige Beispiele.

1. Fehlerabschätzungen im Inneren des Grundgebietes

1.1. Differentialgleichungen mit konstanten Koeffizienten

Mit G sei ein offenes, zusammenhängendes, beschränktes Gebiet im $x_1, \ldots, x_n$-Raum bezeichnet, mit Γ der Rand von G. Für das n-tupel $p_1, \ldots, p_n$ wird auch kurz p oder P geschrieben (gelegentlich p als Spaltenvektor aufgefaßt).

Der Abstand eines Punktes P vom Rande Γ sei mit $R(P)$ bezeichnet, d.h.

$$R(P) = \operatorname*{Min}_{Q \in \Gamma} |P - Q| = \operatorname*{Min}_{Q \in \Gamma} \sqrt{\sum_{j=1}^{n} (p_j - q_j)^2}.$$

R sei der Radius der größten Hyperkugel in $G + \Gamma$, d.h. $R = \sup\limits_{P \in G} R(P)$. Zunächst sei mit dem Laplaceschen Operator $\Delta = \sum\limits_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ und konstantem $c \geqq 0$ der spezielle lineare Differentialausdruck

$$L_1[u] \equiv -\Delta u + c\,u \tag{1.1}$$

betrachtet.

Satz 1. *Sei $u(P)$ eine in $G+\Gamma$ stetige Funktion, die in G stetige partielle Ableitungen bis zur zweiten Ordnung einschließlich hat. Sei $L_1[u]=F(P)$ in G mit einer Funktion $F(P)$, die in G stetig und beschränkt ist und dort stetige partielle Ableitungen erster Ordnung hat. Sei P_0 ein Punkt aus G und für $r \leqq R(P_0)$*

Ω_r die Menge der Punkte Q mit $|Q-P_0|=r$,
K_r die Menge der Punkte Q mit $|Q-P_0|<r$,
$\bar{K}_r$ die Menge der Punkte Q mit $|Q-P_0|\leqq r$.

Dann gilt

$$|u_\xi(P_0)| \leqq A_1 \cdot r \cdot \operatorname*{Max}_{\bar{K}_r} |F(Q)| + A_2 \cdot \frac{1}{r} \cdot \operatorname*{Max}_{\Omega_r} |u(Q)|, \tag{1.2}$$

$$|u_\xi(P_0)| \leqq A_1^* \cdot r^2 \cdot \operatorname*{Max}_{\bar{K}_r} \left|\frac{F(Q)-F(P_0)}{Q-P_0}\right| + A_2 \cdot \frac{1}{r} \cdot \operatorname*{Max}_{\Omega_r} |u(Q)|, \tag{1.3}$$

$$\begin{aligned} |u_{\xi\eta}(P_0)| \leqq {} & \big(|\cos\omega|\, B_{11} + |\sin\omega|\, B_{12}\big) \cdot r \cdot \operatorname*{Max}_{\bar{K}_r} \left|\frac{F(Q)-F(P_0)}{Q-P_0}\right| + \\ & + \big(|\cos\omega|\, B_{21} + |\sin\omega| B_{22}\big) \cdot \frac{1}{r^2} \cdot \operatorname*{Max}_{\Omega_r} |u(Q)| + \\ & + \frac{|\cos\omega|}{n} \cdot \big[|F(P_0)| + c\,|u(P_0)|\big]. \end{aligned} \tag{1.4}$$

Dabei sind ξ und η beliebig vorgegebene Einheitsvektoren, u_ξ, $u_{\xi\eta}$ die Ableitungen von u in den entsprechenden Richtungen, $\cos\omega = (\xi,\eta) = \sum_{j=1}^{n} \xi_j\,\eta_j$ das innere Produkt von ξ und η. Die Konstanten A_i, A_1^*, B_{ij} hängen noch von der Dimension n ab und sind in der Tabelle 1 angegeben.

Bemerkungen zu Satz 1. Die Abschätzungen vereinfachen sich etwas, wenn man die auftretenden Maxima durch die entsprechenden oberen Grenzen über ganz G ersetzt, $|\operatorname{grad} F(Q)|$ für $\left|\frac{F(Q)-F(P_0)}{Q-P_0}\right|$ setzt und $r=R(P_0)$ wählt. Außerdem kann in (1.4) gesetzt werden:

$$\big(|\cos\omega|\, B_{11} + |\sin\omega|\, B_{12}\big) \leqq B_1 = \sqrt{B_{11}^2 + B_{12}^2}$$

und entsprechend

$$\big(|\cos\omega|\, B_{21} + |\sin\omega|\, B_{22}\big) \leqq B_2 = \sqrt{B_{21}^2 + B_{22}^2}.$$

In Tabelle 2 sind für die Koeffizienten von (1.2) bis (1.4) im Falle $c>0$, $n=2$ und $n=3$ Formeln angegeben, die außer von n auch noch von c und r abhängen und insbesondere für große c oder große r bessere Schranken liefern. Für die Funktion u selbst werden Schranken als bekannt vorausgesetzt. Diese seien etwa durch Anwendung der Methoden von [*9*] gewonnen.

Trotz des Faktors $1/r$ beim zweiten Glied der Formeln (1.2) und (1.3) läßt sich in gewissen Fällen auch noch am Rande eine Abschätzung mit diesen Formeln durchführen, nämlich immer dann, wenn für $u(Q)$ eine Abschätzung vorliegt, die genügend stark verschwindet für $R(Q)\to 0$ (s. Beispiel 3.2).

Beweis von Satz 1. P_0 ist ein beliebiger, für den Beweis fester Punkt aus G. Einfachheitshalber sei der Koordinatenanfangspunkt nach P_0 gelegt, also $P_0=(0,\ldots,0)$. Außerdem sei das Koordinatensystem so gedreht, daß der Vektor ξ in die Richtung der positiven x_1-Achse weist und η in der x_1, x_2-Ebene liegt, d. h. $\xi - (1, 0, \ldots, 0)$, $\eta = (\cos\omega, \sin\omega, 0, \ldots, 0)$.

Sei

$$K(n)=\left(\int_0^\pi \sin^{n-2}\alpha\,d\alpha\right)^{-1}=\begin{cases}\frac{1}{\pi}\ \text{für } n=2, \quad \frac{1}{2}\ \text{für } n=3\\ \frac{2\cdot 4\ldots(n-2)}{3\cdot 5\ldots(n-3)}\cdot\frac{1}{\pi}\ \text{für } n>2 \text{ gerade}\\ \frac{3\cdot 5\ldots(n-2)}{2\cdot 4\ldots(n-3)}\cdot\frac{1}{2}\ \text{für } n>3 \text{ ungerade}\end{cases}$$

Tabelle 1

	$n=2$	$n=3$	allgemein
A_1	$\frac{4}{3\pi}$	$\frac{3}{8}$	$\frac{2n}{(n-1)(n+1)}K(n)$
A_2	$\frac{4}{\pi}$	$\frac{3}{2}$	$\frac{2n}{(n-1)}K(n)$
A_1^*	$\frac{1}{2\pi}$	$\frac{3}{20}$	$\frac{n}{(n-1)(n+2)}K(n)$
B_{11}	$\frac{8}{5\pi}$	$\frac{10\sqrt{3}}{27}$	$\sqrt{\frac{(n-1)^{n-1}}{n^n}}\,\frac{n+2}{n+3}\,4K(n)$
B_{12}	$\frac{8}{5\pi}$	$\frac{5}{3\pi}$	$\frac{n+2}{n+3}\,\frac{2}{\pi}$
B_{21}	$\frac{8}{\pi}$	$\frac{20\sqrt{3}}{9}$	$\sqrt{\frac{(n-1)^{n-1}}{n^n}}\,(n+2)\,4K(n)$
B_{22}	$\frac{8}{\pi}$	$\frac{10}{\pi}$	$(n+2)\cdot\frac{2}{\pi}$

Tabelle 2 (mit $\varrho=\sqrt{c}\cdot r$)

	$n=2$	$n=3$
A_1	$\frac{1}{\varrho}\int_0^\varrho\left\{-\xi H_1(i\xi)+\xi J_1(i\xi)\,\frac{H_1(i\varrho)}{J_1(i\varrho)}\right\}d\xi$	$\frac{3}{8}\cdot\frac{8[\varrho(\sinh\varrho-\varrho)-(\cosh\varrho-1-\frac{1}{2}\varrho^2)]}{3\varrho(\varrho\cosh\varrho-\sinh\varrho)}$
A_2	$\frac{4}{\pi}\cdot\left(\frac{-\varrho}{2i\,J_1(i\varrho)}\right)$	$\frac{3}{2}\cdot\frac{\varrho^3}{3(\varrho\cosh\varrho-\sinh\varrho)}$
A_1^*	$\frac{1}{2\pi}\cdot\frac{4\varrho+8i\,J_1(i\varrho)}{\varrho^2 i\,J_1(i\varrho)}$	$\frac{3}{20}\cdot\frac{10[\varrho\cosh\varrho-\sinh\varrho-\frac{1}{3}\varrho^3]}{\varrho^2(\varrho\cosh\varrho-\sinh\varrho)}$
B_{12}	$=B_{11}$ (s. u.)	$=\frac{3\sqrt{3}}{2\pi}B_{11}$ (s. u.)
B_{21}	$\frac{8}{\pi}\cdot\frac{\varrho^2}{8\left[J_0(i\varrho)+\frac{2}{\varrho}i\,J_1(i\varrho)\right]}$	$\frac{20\sqrt{3}}{9}\cdot\frac{\varrho^5}{15[\varrho^2\sinh\varrho-3\varrho\cosh\varrho+3\sinh\varrho]}$
B_{22}	$=B_{21}$ (s. o.)	$\frac{3\sqrt{3}}{2\pi}\cdot B_{21}$ (s. o.)

$$n=2:\quad B_{11}=\frac{1}{2\varrho}\int_0^\varrho\xi^2\left\{i\,H_0(i\xi)-\frac{2}{\xi}H_1(i\xi)-\left[J_0(i\xi)+\frac{2}{\xi}i\,J_1(i\xi)\right]\frac{i\,H_0(i\varrho)-\frac{2}{\varrho}H_1(i\varrho)}{J_0(i\varrho)+\frac{2}{\varrho}i\,J_1(i\varrho)}\right\}d\xi$$

$$n=3:\quad B_{11}=\frac{10\sqrt{3}}{27}\cdot\frac{16\varrho^2\cosh\varrho-48\varrho\sinh\varrho+48\cosh\varrho-2\varrho^4+8\varrho^2-48}{5\varrho^3\sinh\varrho-15\varrho^2\cosh\varrho+15\varrho\sinh\varrho}$$

Zu L_1 und K_r existiert die Greensche Funktion $G(P, Q)$, es ist in K_r

$$u(P) = \int_{K_r} F(Q) \cdot G(P, Q) \cdot dV_Q - \int_{\Omega_r} u(Q) \cdot G_{n_Q}(P, Q) \cdot dS_Q,$$

wo dV_Q das Volumelement in K_r, n_Q die äußere Normale und dS_Q das Oberflächenelement auf Ω_r ist. Mit Berücksichtigung des Verhaltens für $P = Q$ (E. Hopf [7], §2) erhält man durch Differentiation:

$$(1.2^*) \qquad u_\xi(P) = \int_{K_r} F(Q) \cdot G_{\xi_P}(P, Q) \cdot dV_Q - \int_{\Omega_r} u(Q) \cdot G_{n_Q \xi_P}(P, Q) \cdot dS_Q,$$

$$(1.3^*) \qquad \begin{aligned} u_\xi(P) = {} & F(P) \cdot \Big[\int_{K_r} G(P, Q) \cdot dV_Q \Big]_\xi + \\ & + \int_{K_r} [F(Q) - F(P)] \cdot G_{\xi_P}(P, Q) \cdot dV_Q - \int_{\Omega_r} u(Q) \cdot G_{n_Q \xi_P}(P, Q) \cdot dS_Q, \end{aligned}$$

$$(1.4^*) \qquad \begin{aligned} u_{\xi\eta}(P) = {} & F(P) \cdot \Big[\int_{K_r} G(P, Q) \cdot dV_Q \Big]_{\xi\eta} + \\ & + \int_{K_r} [F(Q) - F(P)] \cdot G_{\xi_P \eta_P}(P, Q) \cdot dV_Q - \int_{\Omega_r} u(Q) \cdot G_{n_Q \xi_P \eta_P}(P, Q) \cdot dS_Q. \end{aligned}$$

Die angehängten Indizes P, Q sollen angeben, bezüglich welcher Koordinaten die Ableitung zu nehmen ist.

Nun ist $\varphi(P) = \int_{K_r} G(P, Q) \cdot dV_Q$ kugelsymmetrisch bezüglich P_0, also $\varphi_j(P_0) = \varphi_{jk}(P_0) = 0$ für $j \neq k$ und $\varphi_{jj}(P_0) = \frac{1}{n} \Delta \varphi(P_0)$; außerdem gilt

$$\Delta \varphi - c \varphi = 1,$$

also

$$\Big[\int_{K_r} G(P_0, Q) \cdot dV_Q \Big]_{\xi\eta} = \frac{\cos\omega}{n} \Big[c \int_{K_r} G(P_0, Q) \cdot dV_Q - 1 \Big].$$

Mit $G^*_{\xi_P \eta_P}(P, Q) = G_{\xi_P \eta_P}(P, Q) - \frac{\cos\omega}{n} c\, G(P, Q)$ erhält man

$$(1.4^{**}) \qquad \begin{aligned} u_{\xi\eta}(P_0) = {} & \int_{K_r} [F(Q) - F(P_0)] \cdot G^*_{\xi_P \eta_P}(P_0, Q) \cdot dV_Q - \\ & - \int_{\Omega_r} u(Q) \cdot G^*_{n_Q \xi_P \eta_P}(P_0, Q) \cdot dS_Q + \frac{\cos\omega}{n} [c\, u(P_0) - F(P_0)]. \end{aligned}$$

Aus der Abschätzung der Darstellungen (1.2*), (1.3*) und (1.4**) ergeben sich für $P = P_0$ die Formeln (1.2) bis (1.4), wenn man die A_i, A_1^*, B_{ij} so wählt, daß gilt:

$$A_1 \geqq \int_{K_r} \frac{1}{r} \, |G_{\xi_P}(P_0, Q)| \cdot dV_Q, \qquad A_2 \geqq \int_{\Omega_r} r \, |G_{n_Q \xi_P}(P_0, Q)| \cdot dS_Q,$$

$$A_1^* \geqq \int_{K_r} \frac{|Q - P_0|}{r^2} \cdot |G_{\xi_P}(P_0, Q)| \cdot dV_Q,$$

$$|\cos\omega|\, B_{11} + |\sin\omega|\, B_{12} \geqq \int_{K_r} \frac{|Q - P_0|}{r} \cdot |G^*_{\xi_P \eta_P}(P_0, Q)| \cdot dV_Q,$$

$$|\cos\omega|\, B_{21} + |\sin\omega|\, B_{22} \geqq \int_{\Omega_r} r^2 \cdot |G^*_{n_Q \xi_P \eta_P}(P_0, Q)| \cdot dS_Q.$$

Zur Auswertung dieser Integrale werden zunächst die benötigten Greenschen Funktionen und ihre Ableitungen aufgestellt.

Die sog. Hauptlösung $\gamma_n^c(a)$ hängt nur vom Abstand $a=\sqrt{\sum (x_i-y_j)^2}$ der Punkte $P=(x_1,\ldots,x_n)$ und $Q=(y_1,\ldots,y_n)$ ab:

$$\gamma_n^c(a)=\begin{cases} -\frac{1}{2\pi}\cdot\ln a & \text{für } n=2 \\ \frac{1}{(n-2)\cdot\omega_n}\cdot a^{-(n-2)}, \qquad \omega_n=\frac{2\sqrt{\pi}^n}{\Gamma(n/2)} & \text{für } n>2 \end{cases}\Bigg\} \; c=0,$$

$$\gamma_n^c(a)=\begin{cases} \frac{i}{4}\cdot H_0^{(1)}(i\sqrt{c}\,a) & \text{für } n=2 \\ \frac{i}{4}\cdot\left(\frac{i\sqrt{c}}{2\pi}\right)^{\frac{n-2}{2}}\cdot a^{-\frac{n-2}{2}}\cdot H^{(1)}_{\frac{n-2}{2}}(i\sqrt{c}\,a) & \text{für } n>2 \text{ gerade} \\ -\frac{i}{4}\cdot\left(-\frac{i\sqrt{c}}{2\pi}\right)^{\frac{n-2}{2}}\cdot a^{-\frac{n-2}{2}}\cdot J_{-\frac{n-2}{2}}(i\sqrt{c}\,a) & \text{für } n>2 \text{ ungerade} \end{cases}\Bigg\} \; c>0.$$

Als reguläre kugelsymmetrische Lösung bietet sich an:

$$g_n^c(a)=\begin{cases} 1 & \text{für } c=0, \\ \left(-\frac{i\sqrt{c}}{2\pi}\right)^{\frac{n-2}{2}}\cdot a^{-\frac{n-2}{2}}\cdot J_{\frac{n-2}{2}}(i\sqrt{c}\,a) & \text{für } c>0. \end{cases}$$

Im folgenden wird auch bei der Greenschen Funktion immer noch die Dimension als Index mit angegeben. Wenn der Index $n+2$ auftritt, soll die entsprechende Funktion des R_{n+2} betrachtet werden für $x_{n+1}=x_{n+2}=y_{n+1}=y_{n+2}=0$. Entsprechendes gilt für $n+4$ und $n+6$. Der Ansatz

$$G_n(P_0,Q)=\gamma_n^c(\varrho)-\frac{\gamma_n^c(r)}{g_n^c(r)}\,g_n(\varrho)+h(P_0,Q)$$

mit $\varrho=|P_0-Q|=\sqrt{\sum y_j^2}$ führt auf $h(P_0,Q)=0$ in K_r als eindeutige Lösung der Randwertaufgabe $-\Delta_Q h(P_0,Q)+c\,h(P_0,Q)=0$ in K_r, $h(P_0,Q)=0$ auf Ω_r, so daß gilt

$$G_n(P_0,Q)=\gamma_n^c(\varrho)-\frac{\gamma_n^c(r)}{g_n^c(r)}\,g_n^c(\varrho).$$

Wegen

$$\gamma^c_{n\,\xi_P}(a)=-2\pi(x_1-y_1)\cdot\gamma^c_{n+2}(a),$$

$$\gamma^c_{n\,\xi_P\eta_P}(a)=\cos\omega\cdot\frac{c}{n}\cdot\gamma_n^c(a)+$$
$$+4\pi^2\left\{\cos\omega\cdot\left[(x_1-y_1)^2-\frac{a^2}{n}\right]+\sin\omega\cdot(x_1-y_1)(x_2-y_2)\right\}\cdot\gamma^c_{n+4}(a)$$

erhält man entsprechend

$$G_{n\,\xi_P}(P_0,Q)=2\pi\,y_1\,G_{n+2}(P_0,Q)$$

$$G_{n\,\xi_P\eta_P}(P_0,Q)=4\pi^2\left[\cos\omega\cdot\left(y_1^2-\frac{\varrho^2}{n}\right)+\sin\omega\cdot y_1\cdot y_2\right]G_{n+4}(P_0,Q)$$

und durch Ableitung nach ϱ und $\varrho \to r$ auf dem Rande Ω_r:

$$G_{n\,n_Q\,\xi_P}(P_0, Q) = \quad 4\pi^2 r\, y_1 \left[\gamma^c_{n+4}(r) + \begin{cases} 0 & \\ \dfrac{\gamma^c_{n+2}(r)}{\xi^c_{n+2}(r)}\, g^c_{n+4}(r) & \end{cases}\right] \begin{matrix} \text{für } c = 0 \\ \text{für } c > 0, \end{matrix}$$

$$G_{n\,n_Q\,\xi_P\,\eta_P}(P_0, Q) = -8\pi^3 r \left[\cos\omega \cdot \left(y_1^2 - \frac{r^2}{n}\right) + \sin\omega\, y_1 y_2\right] \times$$

$$\times \left[\gamma^c_{n+6}(r) + \begin{cases} 0 & \\ \dfrac{\gamma^c_{n+4}(r)}{\xi^c_{n+4}(r)}\, g^c_{n+6}(r) & \end{cases}\right] \begin{matrix} \text{für } c = 0 \\ \text{für } c > 0. \end{matrix}$$

Damit lassen sich alle Integranden explizit angeben. Ihre Integration ergibt für $c=0$ die Tabelle 1. Für $c>0$ ist die Integration teilweise nur angenähert möglich. In Tabelle 2 sind für $n=2$ und $n=3$ die Ergebnisse, soweit elementar angebbar, zusammengestellt. Die Schranken werden kleiner bei wachsendem c. Einfacher kann man auch für $c>0$ durch die Formeln (1.2) bis (1.4) unter Benutzung von Tabelle 1 abschätzen, wie sich aus den Ungleichungen

$$\begin{aligned} |G^c_{n+i}(P_0, Q)| &\leqq |G^0_{n+i}(P_0, Q)| \\ \left|\frac{\partial}{\partial\varrho}\, G^c_{n+i}(P_0, Q)\right|_{\varrho=r} &\leqq \left|\frac{\partial}{\partial\varrho}\, G^0_{n+i}(P_0, Q)\right|_{\varrho=r} \end{aligned} \qquad i = 2, 4$$

ergibt, wobei der hochgestellte Index c bzw. 0 angeben soll, daß die Greensche Funktion für den Differentialausdruck $-\Delta G + c\,G$ zu nehmen ist mit $c>0$ bzw. $c=0$.

Die zweite Ungleichung ist eine direkte Folge der ersten, die erste folgt aus der Monotonie der Randwertaufgabe ([*9*], Monotoniesatz 1) mit dem Laplace-Operator, denn es ist

$$-\Delta_Q\, G^c_{n+i}(P_0, Q) = -c\, G^c_{n+i}(P_0, Q) \leqq 0 = -\Delta_Q\, G^0_{n+i}(P_0, Q) \quad \text{in } K_r,$$

$G^c_{n+i}(P_0, Q) = G^0_{n+i}(P_0, Q) = 0$ auf Ω_r und für kleines ϱ ist $G^c_{n+i}(P_0, Q) \leqq G^0_{n+i}(P_0, Q)$, wie man leicht aus der Reihenentwicklung der Zylinderfunktionen [*3*], (Bd. 1, S. 420, 432) sieht.

Damit ist Satz 1 bewiesen.

Nun sei allgemeiner in G ein linearer Differentialausdruck

$$L[u] \equiv -\sum_{k,l=1}^{n} a_{kl}\, u_{kl} - \sum_{j=1}^{n} b_j\, u_j + c\, u \tag{1.5}$$

vorgelegt mit konstanten reellen Koeffizienten a_{kl}, b_j, c. Bei der Funktion u geben die Indizes die betreffenden partiellen Ableitungen an. Der Differentialausdruck erfülle die Voraussetzungen:

(1.6) Die Matrix $A = (a_{kl})$ ist symmetrisch und positiv definit. Es ist $c \geqq 0$.

Die Eigenwerte λ_i von A (d.h. die Wurzeln von $\det(A - \lambda E) = 0$) sind also positiv, es gibt zwei Zahlen l_u und l_0, so daß

$$0 < l_u \leqq \sqrt{\lambda_i} \leqq l_0 \qquad \text{ist für } i = 1, \ldots, n. \tag{1.7}$$

Mit den Koeffizienten b_j von (1.5) und l_u, l_0 wird gebildet

$$\alpha = \frac{|b|}{2 l_u l_0}, \qquad |b| = \sqrt{\sum_{j=1}^{n} b_j^2}. \tag{1.8}$$

Die Differentialgleichung $L[u]=f(x)$ für $u(x)$ läßt sich reduzieren in eine Differentialgleichung $L_1[v]=F(\tilde{x})$ für eine Hilfsfunktion $v(\tilde{x})$ ([3], Bd. 2, S. 137), auf welche die Abschätzungen nach Satz 1 angewandt werden können:

Sei T die orthogonale Matrix der Transformation von A auf Hauptachsen, D die Diagonalmatrix mit den Elementen $d_{ii}=\sqrt{\lambda_i}$. Durch die affine Transformation $x=TD\tilde{x}$ geht $L[u]=f(x)$ über in $\tilde{L}[\tilde{u}]=\tilde{f}(\tilde{x})$ mit $\tilde{L}[\tilde{u}]\equiv-\Delta\tilde{u}-\sum_{j=1}^{n}\tilde{b}_j\tilde{u}_j+c\tilde{u}$, $\tilde{u}(\tilde{x})=u(x)$, $\tilde{f}(\tilde{x})=f(x)$. Im $\tilde{x}$-Raum wird die Hilfsfunktion $v(\tilde{x})=e^{\frac{1}{2}(\tilde{b},\tilde{x})}\tilde{u}(\tilde{x})$ eingeführt mit $(\tilde{b},\tilde{x})=\sum_{j=1}^{n}\tilde{b}_j\tilde{x}_j$. Aus $\tilde{L}[\tilde{u}]=\tilde{f}(\tilde{x})$ wird dann die Differentialgleichung

$$\tilde{L}_1[v]\equiv-\Delta v+\tilde{c}\,v=F(\tilde{x})\quad\text{mit}\quad\tilde{c}=c+\tfrac{1}{4}|\tilde{b}|^2,\qquad F(\tilde{x})=\tilde{f}(\tilde{x})\,e^{\frac{1}{2}(\tilde{b},\tilde{x})}.$$

Die Formeln (1.2) bis (1.4) von Satz 1 liefern Schranken für die Ableitungen von $v(\tilde{x})$. Rücktransformation auf die Funktion $u(x)$ im x-Raum ergibt den

Satz 2. *Sei $u(P)$ eine in $G+\Gamma$ stetige Funktion, die in G noch stetige partielle Ableitungen bis zur zweiten Ordnung einschließlich hat. Sei $L[u]=f(P)$ in G mit einer Funktion $f(P)$, die in G stetig und beschränkt ist und dort stetige partielle Ableitungen erster Ordnung hat.*

Dann gelten im Punkt $P_0\in G$ die Abschätzungen:

$$|u_\xi(P_0)|\leq A_1\cdot r\cdot\frac{e^{\alpha r}}{l_u\,l_0}\,\underset{\bar{K}_r}{\operatorname{Max}}|f(Q)|+\frac{A_2}{r}\,\frac{l_0\,e^{\alpha r}}{l_u}\,\underset{\bar{K}_r}{\operatorname{Max}}|u(Q)|+\alpha\,\frac{l_0}{l_u}\cdot|u(P_0)|,\tag{1.9}$$

$$\begin{aligned}|u_\xi(P_0)|\leqq{}&A_1^*\,r^2\,\frac{e^{\alpha r}}{l_u\,l_0}\left[\underset{\bar{K}_r}{\operatorname{Max}}\left|\frac{f(Q)-f(P_0)}{Q-P_0}\right|+\alpha\,|f(P_0)|\right]+\\&+\frac{A_2}{r}\,\frac{l_0\,e^{\alpha r}}{l_u}\,\underset{\bar{K}_r}{\operatorname{Max}}|u(Q)|+\alpha\,\frac{l_0}{l_u}\,|u(P_0)|,\end{aligned}\tag{1.10}$$

$$\begin{aligned}|u_{\xi\eta}(P_0)|\leqq{}&(B_1r+2\alpha\,A_1^*\,r^2)\,\frac{e^{\alpha r}}{l_u^2}\left[\underset{\bar{K}_r}{\operatorname{Max}}\left|\frac{f(Q)-f(P_0)}{Q-P_0}\right|+\alpha\cdot|f(P_0)|\right]+\\&+\left(\frac{B_2}{r^2}+2\alpha\,\frac{A_2}{r}\right)\frac{e^{\alpha r}\,l_0^2}{l_u^2}\,\underset{\bar{K}_r}{\operatorname{Max}}|u(Q)|+\\&+\frac{1}{n\,l_u^2}\left[|f(P_0)|+(c+(n+1)\,\alpha^2\,l_0^2)\,|u(P_0)|\right]\end{aligned}\tag{1.11}$$

für $r\leqq R(P_0)$.

Dabei sind ξ und η wieder beliebig vorgegebene Einheitsvektoren. A_i, A_1^*, B_i sind die Konstanten der Tabelle 1 mit $B_1=\sqrt{B_{11}^2+B_{12}^2}$, $B_2=\sqrt{B_{21}^2+B_{22}^2}$.

Die anschließend an Satz 1 gemachten Bemerkungen können sinngemäß hierher übertragen werden.

1.2. Differentialgleichungen mit veränderlichen Koeffizienten

In G sei ein linearer Differentialausdruck

$$L[u]=-\sum_{j,k=1}^{n}a_{jk}(P)\cdot u_{jk}(P)-\sum_{j=1}^{n}b_j(P)\cdot u_j(P)+c(P)\cdot u(P)\tag{1.12}$$

vorgelegt. Er erfülle die Voraussetzungen

(1.13) Die Koeffizienten a_{jk}, b_j, c sind stetig und mit stetigen Ableitungen erster Ordnung versehen in $G+\Gamma$. Die Matrix A der $a_{jk}(P)$ ist symmetrisch und positiv definit in jedem Punkt $P\in G+\Gamma$ und es ist $c(P)\geqq0$ dort.

Die Eigenwerte $\lambda_i(P)$ der Matrix $A(P)$ sind also positiv für $P \in G+\Gamma$. Wegen Stetigkeit im abgeschlossenen Bereich $G+\Gamma$ gibt es analog zu (1.7) zwei Zahlen l_u und l_0, so daß gilt

$$(1.14) \qquad 0 < l_u \leqq \sqrt{\lambda_i(P)} \leqq l_0 \quad \text{für} \quad i = 1, \dots, n \quad \text{und} \quad P \in G+\Gamma.$$

Analog zu (1.8) wird gebildet

$$(1.15) \qquad \alpha = \frac{1}{2 l_u l_0} \operatorname*{Max}_{G+\Gamma} |b(P)| \quad \text{mit} \quad |b(P)| = \sqrt{\sum_{j=1}^{n} b_j^2(P)}.$$

Zu den Koeffizienten von (1.12) werden Konstanten c, A, B, C gebildet mit

$$(1.16) \qquad \begin{aligned} & c \geqq \operatorname*{Max}_{P \in G+\Gamma} c(P), \\ & A \geqq \operatorname*{Max}_{\substack{P \in G+\Gamma \\ Q \in G+\Gamma}} \left[\sum_{j,k=1}^{n} \left| \frac{a_{jk}(Q) - a_{jk}(P)}{Q-P} \right| \right] && \left(\text{etwa } A = \sum_{j,k=1}^{n} \operatorname*{Max}_{G+\Gamma} |\operatorname{grad} a_{jk}| \right), \\ & B \geq \operatorname*{Max}_{\substack{P \in G+\Gamma \\ Q \in G+\Gamma}} \left[\sum_{j=1}^{n} \left| \frac{b_j(Q) - b_j(P)}{Q-P} \right| \right] && \left(\text{etwa } B = \sum_{j=1}^{n} \operatorname*{Max}_{G+\Gamma} |\operatorname{grad} b_j| \right), \\ & C \geqq \operatorname*{Max}_{\substack{P \in G+\Gamma \\ Q \in G+\Gamma}} \left| \frac{c(Q) - c(P)}{Q-P} \right| && \left(\text{etwa } C = \operatorname*{Max}_{G+\Gamma} |\operatorname{grad} c| \right). \end{aligned}$$

Seien weiter U, F_0, F_1 Konstanten mit

$$(1.17) \qquad \begin{aligned} & U \geqq \operatorname*{Max}_{P \in G+\Gamma} |u(P)|, \qquad F_0 \geqq \operatorname*{Max}_{P \in G+\Gamma} |F(P)|, \\ & F_1 \geqq \operatorname*{Max}_{\substack{P \in G+\Gamma \\ Q \in G+\Gamma}} \left[\left| \frac{F(Q) - F(P)}{Q-P} \right| + \alpha |F(P)| \right]. \end{aligned}$$

Mit diesen Konstanten und mit den in Tabelle 1 angegebenen A_1, A_1^*, A_2, $B_1 = \sqrt{B_{11}^2 + B_{12}^2}$, $B_2 = \sqrt{B_{21}^2 + B_{22}^2}$ werden Hilfsfunktionen gebildet:

$$(1.18) \qquad \begin{aligned} & h_{10}(\delta) = \delta^2 e^{\alpha R \delta} \frac{A_1^* R^3}{l_u l_0} (F_1 + C U) + \alpha \frac{R l_0}{l_u} U + \frac{e^{\alpha R \delta} A_2 l_0}{\delta l_u} U, \\ & h_{11}(\delta) = \frac{\delta^2 e^{\alpha R \delta}}{1-\delta} \frac{A_1^* \cdot R^2}{l_u l_0} B, \qquad h_{12}(\delta) = \frac{\delta^2 e^{\alpha R \delta}}{(1-\delta)^2} \frac{A_1^* \cdot R}{l_u l_0} A, \\ & h_{20}(\delta) = \delta e^{\alpha R \delta} (B_1 + 2\alpha A_1^* R \delta) \frac{R^3}{l_u^2} (F_1 + C U) + \frac{R^2 [F_0 + (c + (n+1) \alpha^2 l_0^2) U]}{4 n l_u^2} + \\ & \qquad + \frac{e^{\alpha R \delta}}{\delta^2} (B_2 + 2\alpha A_2 R \delta) \frac{l_0^2}{l_u^2} U, \\ & h_{21}(\delta) = \frac{\delta}{1-\delta} e^{\alpha R \delta} (B_1 + 2\alpha A_1^* R \delta) \frac{R^2}{l_u^2} B, \\ & h_{22}(\delta) = \frac{\delta}{(1-\delta)^2} e^{\alpha R \delta} (B_1 + 2\alpha A_1^* R \delta) \frac{R}{l_u^2} A. \end{aligned}$$

Alle diese Funktionen sind positiv und stetig in offenen Intervall $0 < \delta < 1$. Für $\delta \to 1$ wachsen die Funktionen $h_{ij}(\delta)$ mit $i, j = 1, 2$ über alle Grenzen. Für $\delta \to 0$ gilt

$$\lim_{\delta \to 0} h_{ij}(\delta) = 0 \quad \text{für} \quad \begin{cases} i = 1, 2 \\ j = 1, 2. \end{cases}$$

Zu vorgegebenem $k>0$ gibt es also ein $\varepsilon>0$, so daß $h_{ij}(\delta)<k$ ist für $0<\delta<\varepsilon$ und $i, j=1, 2$.

Damit kann der Abschätzungssatz formuliert werden:

Satz 3. *Sei $u(P)$ eine in $G+\Gamma$ stetige Funktion mit stetigen partiellen Ableitungen bis zur zweiten Ordnung einschließlich in G. Mit dem in* (1.12) *definierten Differentialausdruck L sei $L[u]=F(P)$ in G beschränkt, stetig und mit stetigen partiellen Ableitungen erster Ordnung versehen. Unter Benutzung der Bezeichnungen von* (1.14) *bis* (1.18) *seien δ_1 und δ_2 zwei Zahlen mit $0<\delta_1, \delta_2<1$ und*

$$h_{11}(\delta_1)<1, \quad h_{22}(\delta_2)<1, \quad \frac{h_{12}(\delta_1)}{1-h_{11}(\delta_1)}\cdot\frac{h_{21}(\delta_2)}{1-h_{22}(\delta_2)}<1.$$

Dann ist für beliebige Richtungen ξ, η

$$\begin{aligned} R(P_0)\cdot|u_\xi(P_0)| &\leqq \frac{(1-h_{22})\,h_{10}+h_{12}\,h_{20}}{(1-h_{11})\,(1-h_{22})-h_{12}\,h_{21}}, \\ R^2(P_0)\cdot|u_{\xi\eta}(P_0)| &\leqq \frac{(1-h_{11})\,h_{20}+h_{21}\,h_{10}}{(1-h_{11})\,(1-h_{22})-h_{12}\,h_{21}}. \end{aligned} \tag{1.19}$$

Dabei ist $P_0\in G$ beliebig. Für $i=1, 2$; $j=0, 1, 2$ ist kurz $h_{ij}(\delta_i)=h_{ij}$ gesetzt worden.

Für eine möglichst gute Abschätzung kann es nützlich sein, als G nicht das Gebiet der vorliegenden Randwertaufgabe, sondern das Innere einer Hyperkugel um P_0 in diesem Gebiet zu nehmen.

Beweis von Satz 3. P_0 ist ein beliebiger, für den Beweis fester Punkt von G. G^* sei das abgeschlossene Teilgebiet $|P-P_0|\leqq dR(P_0)$ mit $0<d<1$ von G, $R^*(P)$ der Abstand des Punktes P vom Rande von G^*. Die Ableitungen von u sind in G^* beschränkt. Sei

$$N_1^*=\underset{\substack{P\in G^*\\ |\varrho_1|=1}}{\operatorname{Max}} R^*(P)\cdot|u_{\varrho_1}(P)|, \qquad N_2^*=\underset{\substack{P\in G^*\\ |\varrho_2|=1\\ |\varrho_3|=1}}{\operatorname{Max}} R^{*2}(P)\cdot|u_{\varrho_2\varrho_3}(P)|, \tag{1.20}$$

wobei zu beachten ist, daß die Größtwerte auch über alle möglichen Richtungen (Einheitsvektoren) $\varrho_1, \varrho_2, \varrho_3$ zu nehmen sind.

Damit gibt es zu $\gamma>1$ im Inneren von G^* Punkte P_1 und P_2 sowie Richtungen $\varrho_1, \varrho_2, \varrho_3$, so daß gilt

$$N_1^*\leqq\gamma\cdot R^*(P_1)\cdot|u_{\varrho_1}(P_1)| \quad \text{und} \quad N_2^*\leqq\gamma\cdot R^{*2}(P_2)\cdot|u_{\varrho_2\varrho_3}(P_2)|.$$

Zu den P_i, $i=1, 2$ werden Differentialgleichungen $L_i^*[u]=f_i(P)$ mit konstanten Koeffizienten gebildet:

$$\begin{aligned} L_i^*[u] &= -\sum_{j,k=1}^{n} a_{jk}(P_i)\,u_{jk}(P) - \sum_{j=1}^{n} b_j(P_i)\,u_j(P) + c(P_i)\,u(P) \\ &= f_i(P) = F(P) + \sum_{j,k=1}^{n}[a_{jk}(P)-a_{jk}(P_i)]\,u_{jk}(P) + \\ &\quad + \sum_{j=1}^{n}[b_j(P)-b_j(P_i)]\,u_j(P) - [c(P)-c(P_i)]\,u(P). \end{aligned}$$

Mit ihrer Hilfe lassen sich die rechten Seiten obiger Ungleichungen weiter abschätzen über die Formeln (1.10) bzw. (1.11). Unter Benutzung der Konstanten von (1.16) und (1.17) und der Schranken N_i^* für die Ableitungen von u erhält man für $\gamma \to 1$ die Ungleichungen

$$N_1^* \leqq h_{10}(\delta_1) + h_{11}(\delta_1) \cdot N_1^* + h_{12}(\delta_1) \cdot N_2^*$$
$$N_2^* \leqq h_{20}(\delta_2) + h_{21}(\delta_2) \cdot N_1^* + h_{22}(\delta_2) \cdot N_2^*$$

mit den Funktionen h_{ij} von (1.18) und $r_i = \delta_i R^*(P_i)$, $0 < \delta_i < 1$. Nach Voraussetzung wurden δ_1 und δ_2 so gewählt, daß diese Ungleichungen aufgelöst werden können, man erhält

$$N_1^* \leqq \frac{(1-h_{22})\, h_{10} + h_{12}\, h_{20}}{(1-h_{11})(1-h_{22}) - h_{12}\, h_{21}}, \qquad N_2^* \leqq \frac{(1-h_{11})\, h_{20} + h_{21}\, h_{10}}{(1-h_{11})(1-h_{22}) - h_{12}\, h_{21}}.$$

Die rechten Seiten sind unabhängig von d. $d \to 1$ liefert die Behauptung des Satzes.

Aus Satz 3 folgt insbesondere, daß $R(P) \cdot |u_\xi(P)|$ und $R^2(P) \cdot |u_{\xi\eta}(P)|$ beschränkt sind in $G + \Gamma$.

2. Fehlerabschätzungen für die Tangentialableitung auf dem Rande des Gebietes bei ebenen Aufgaben

Sei G ein offenes, beschränktes, einfach zusammenhängendes Gebiet in der x, y-Ebene mit stückweise glattem Rand Γ. In G sei der selbstadjungierte Differentialausdruck

(2.1) $$L[u] \equiv -(a\, u_x - b\, u_y)_x - (b\, u_x + c\, u_y)_y$$

vorgelegt mit Funktionen $a(x, y)$, $b(x, y)$ und $c(x, y)$, die in $G + \Gamma$ stetig sind und dort stetige partielle Ableitungen erster Ordnung haben. Die Matrix $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ sei positiv definit in $G + \Gamma$, d.h. es sei $a > 0$ und $D = ac - b^2 > 0$ in $G + \Gamma$.

Von den Funktionen, auf die der Differentialausdruck (2.1) angewandt wird, sei vorausgesetzt, daß sie in G stetige partielle Ableitungen zweiter Ordnung haben, in $G + \Gamma$ stetig sind und stetige partielle Ableitungen erster Ordnung haben.

Γ habe die Darstellung $\big(x(s), y(s)\big)$, $0 \leqq s \leqq l$ mit der Bogenlänge s als Parameter und stückweise stetig differenzierbaren Funktionen $x(s)$, $y(s)$ und $\dot{x}^2 + \dot{y}^2 > 0$ in $0 \leqq s \leqq l$. Γ werde von s im mathematisch positiven Sinn durchlaufen.

Durch $t = (\dot{x}, \dot{y})$ und $\nu = (-\dot{y}, \dot{x})$ sind jedem Punkt von Γ ein Tangentenvektor t und ein innerer Normalvektor ν zugeordnet (eventuell je zwei für dasselbe s als Grenzwerte von kleineren und größeren s her).

Mit der Koeffizientenmatrix A von (2.1) läßt sich auf Γ der sog. Konormalenvektor σ bilden:

(2.2) $$\sigma = \frac{\sigma^*}{|\sigma^*|} \quad \text{mit} \quad \sigma^* = A\nu.$$

Da A positiv definit auf Γ ist, gilt $|\sigma^*| \neq 0$ und $(\sigma, \nu) > 0$. Ableitungen von Funktionen in Richtung der Tangente oder der Konormale werden kurz durch die Indizes t bzw. σ gekennzeichnet.

Bei der zweiten Randwertaufgabe werden oft auf dem Rande Γ die Werte vorgeschrieben für den Differentialausdruck

(2.3) $$L^*[u] = -|\sigma^*|\, u_\sigma = -(a\, u_x + b\, u_y)\, \dot{y} - (b\, u_x + c\, u_y)\, \dot{x}.$$

Satz 4. *Sei* φ *eine Lösung der Differentialgleichung* $L[\varphi]=0$ *in* G. Γ *bestehe aus zwei Kurvenstücken* K_1, K_2 $(\Gamma=K_1+K_2)$, *so daß*

$$L^*[\varphi]\leqq 0 \quad \textit{auf } K_1 \textit{ ist,}$$

$$L^*[\varphi]\geqq 0 \quad \textit{auf } K_2 \textit{ ist.}$$

Dann gilt:

Wenn Γ *glatt ist im Endpunkt* P_1 *von* K_1, *dann ist* $\varphi_t\geqq 0$ *in* P_1,

Wenn Γ *glatt ist im Endpunkt* P_2 *von* K_2, *dann ist* $\varphi_t\leqq 0$ *in* P_2.

Dabei ist unter Kurvenstück das stetige Bild eines Intervalles verstanden. Die Bezeichnung Endpunkt ist im Sinne der Orientierung von Γ durch den Bogenparameter gemeint.

Beweis. Die Hilfsfunktionen

$$f(x,y)=a\,\varphi_x+b\,\varphi_y \quad \text{und} \quad g(x,y)=-b\,\varphi_x-c\,\varphi_y$$

sind eindeutig und stetig in $G+\Gamma$, haben stetige partielle Ableitungen erster Ordnung in G und aus $L[\varphi]=0$ folgt $f_x=g_y$ in G. Nach dem Hauptsatz über Kurvenintegrale ist die Funktion

$$\psi(P)=\psi(P^*)+\int\limits_{P^*}^{P}(g\,dx+f\,dx)$$

mit festem Punkt $P^*\in G$ und vorgegebenem Wert $\psi(P^*)$ in $G+\Gamma$ eindeutig und hat stetige partielle Ableitungen erster Ordnung dort mit $\psi_x=g$, $\psi_y=f$, unabhängig vom gewählten Integrationsweg, solange dieser in $G+\Gamma$ verläuft. $\psi(x,y)$ genügt in G einer elliptischen Differentialgleichung, denn ψ hat auch noch stetige partielle Ableitungen zweiter Ordnung in G und aus $\varphi_{xy}=\varphi_{yx}$ in G folgt

$$-(a\,\psi_x+b\,\psi_y)_x-(b\,\psi_x+c\,\psi_y)_y+\frac{a\,D_x+b\,D_y}{D}\,\psi_x+\frac{b\,D_x+c\,D_y}{D}\,\psi_y=0.$$

Nach dem Randmaximumssatz nimmt also ψ seinen Größtwert und seinen Kleinstwert bezüglich $G+\Gamma$ auf Γ an.

Auf Γ gilt $\psi_t=L^*[\varphi]$, $L^*[\psi]=-D\,\varphi_t$, auf K_2 ist also $\psi_t\geqq 0$, d. h. $\psi(P)-\psi(P_1)\geqq 0$, auf K_1 ist $\psi_t\leqq 0$, d. h. $\psi(P)-\psi(P_1)\geqq 0$, insgesamt $\psi(P)\geqq\psi(P_1)$ auf Γ. Entsprechend erhält man $\psi(P)\leqq\psi(P_2)$ auf Γ. ψ nimmt also seinen Kleinstwert (Größtwert) bezüglich $G+\Gamma$ in P_1 (P_2) an. Wenn Γ glatt ist in P_1, weist die Konormale ins Innere von G bei P_1. Es ist dann also $\psi_\sigma\geqq 0$ in P_1, wegen $D>0$ und $|\sigma^*|>0$ also

$$\varphi_t=-\frac{1}{D}\,L^*[\psi]=\frac{|\sigma^*|}{D}\,\psi_\sigma\geqq 0 \quad \text{in } P_1.$$

Entsprechend folgt

$$\varphi_t=\frac{|\sigma^*|}{D}\,\psi_\sigma\leqq 0 \quad \text{in } P_2,$$

wenn Γ glatt ist dort.

Für die Durchführung einer Fehlerabschätzung bei der Randwertaufgabe

$$L[u]=r(x,y) \quad \text{in } G,$$

$$L^*[u]=\gamma(s) \quad \text{auf } \Gamma$$

sei $v(x, y)$ eine Näherung mit $L[v] = r(x, y)$ in G, $L^*[v] = \gamma(s) + \varepsilon(s)$ auf Γ. Sei weiter $q = \sum_{j=1}^{p} a_j q_j(x, y)$ eine Lösung von $L[q] = 0$ mit noch freien Parametern a_j, $j = 1, \ldots, p$. Auf Γ ist $L^*[q] = \sum_{j=1}^{p} a_j \cdot \gamma_j(s)$ mit $\gamma_j(s) = L^*[q_j]$.

Nun sei P_1 ein Punkt von Γ, bei welchem Γ glatt ist. Zu P_1 gehöre der Wert s_1 der Bogenlänge von Γ. Es müssen Zahlen $a_j^{(1)}$ so bestimmt werden, daß für $q^{(1)} = \sum_{j=1}^{p} a_j^{(1)} \cdot q_j$ auf Γ gilt

$$\begin{aligned} &\varepsilon(s) + \sum_{j=1}^{p} a_j^{(1)} \cdot \gamma_j(s) \geqq 0 \qquad \text{für } s_1 \leqq s \leqq s_2, \\ &\varepsilon(s) + \sum_{j=1}^{p} a_j^{(1)} \cdot \gamma_j(s) \leqq 0 \qquad \text{für } s_2 \leqq s \leqq s_1 + l, \end{aligned} \tag{2.4}$$

wobei $\varepsilon(l+s) = \varepsilon(s)$ und $\gamma_j(l+s) = \gamma_j(s)$ zu setzen ist und s_2 eine Zahl mit $s_1 < s_2 < s_1 + l$ ist. Dann ist nach Satz 4

$$-q_t^{(1)} \leqq v_t - u_t \quad \text{bei } P_1.$$

Um eine möglichst gute Schranke zu erhalten, wird man zu den Ungleichungen (2.4) noch die Extremalbedingung

$$q_t^{(1)}(P_1) = \text{Min} \tag{2.5}$$

hinzunehmen bei der Bestimmung der $a_j^{(1)}$.

Entsprechend müssen Zahlen $a_j^{(2)}$ so bestimmt werden, daß gilt

$$\begin{aligned} &\varepsilon(s) + \sum_{j=1}^{p} a_j^{(2)} \cdot \gamma_j(s) \leqq 0 \qquad \text{für } s_1 \leqq s \leqq s_3, \\ &\varepsilon(s) + \sum_{j=1}^{p} a_j^{(2)} \cdot \gamma_j(s) \geqq 0 \qquad \text{für } s_3 \leqq s \leqq s_1 + l, \end{aligned} \qquad q_t^{(2)}(P_1) = \text{Max},$$

denn damit erhält man $v_t - u_t \leqq -q_t^{(2)}$ bei P_1.

Die Abschätzung mittels Satz 4 kann nur punktweise erfolgen.

In den Anwendungen besonders wichtig sind zweite Randwertaufgaben bei unbeschränktem Grundgebiet.

Sei B ein nichtbeschränkter Bereich der x, y-Ebene, der alle Punkte P mit $r \geqq r_0$ für ein passend gewähltes r_0 enthält, wo $r = \sqrt{x^2 + y^2}$ ist. Der Koordinatenanfangspunkt liege außerhalb B. Der Rand Γ von B bilde eine stückweise glatte, geschlossene, doppelpunktfreie Kurve. Auf Γ sei wieder die Bogenlänge s eingeführt, B liege zur Linken beim Durchlaufen von Γ mit wachsendem s.

Satz 5. *Sei φ eine Lösung der Potentialgleichung $\Delta\varphi = 0$ in B. Es gebe positive Konstanten c und ε, so daß in B gilt*

$$|\text{grad}\, \varphi| \leqq \frac{c}{r^{1+\varepsilon}}.$$

Γ bestehe aus zwei Kurvenstücken K_1, K_2, so daß $-\varphi_\nu \leqq 0$ auf K_1, $-\varphi_\nu \geqq 0$ auf K_2 ist.

Wenn Γ glatt ist im Endpunkt P_1 von K_1, dann ist $\varphi_t \geqq 0$ in P_1,
Wenn Γ glatt ist im Endpunkt P_2 von K_2, dann ist $\varphi_t \leqq 0$ in P_2.

Der Beweis ist sehr einfach. Er wird hier nur angedeutet:

$$\varphi(P) \text{ ist beschränkt für } r \to \infty \text{ wegen } |\operatorname{grad} \varphi| \leqq \frac{c}{r^{1+\varepsilon}}\,.$$

Die Transformation durch reziproke Radien

$$\begin{cases} \xi = \dfrac{x}{r^2}, \\ \eta = \dfrac{y}{r^2}, \end{cases} \qquad \begin{cases} x = \dfrac{\xi}{\varrho^2}, \\ y = \dfrac{\eta}{\varrho^2}, \end{cases} \qquad \text{mit} \quad \varrho^2 = \xi^2 + \eta^2$$

führt den Bereich B in einen beschränkten Bereich B^* der ξ, η-Ebene über. Es ist $\Delta \varphi^* = 0$ in B^* für $\varphi^*(\xi, \eta) = \varphi(x, y)$. Da $\varphi(x, y)$ beschränkt ist für $r \to \infty$, gilt dasselbe für $\varphi^*(\xi, \eta)$ bei $\varrho \to 0$. Dann hat φ^* im Nullpunkt eine hebbare Singularität, kann dort also so definiert werden, daß $\Delta \varphi^* = 0$ ist in dem einfach zusammenhängenden Bereich G^*, der aus B^* durch Hinzunahme des Nullpunktes entsteht ([*6*], S. 453). Nun läßt sich Satz 4 anwenden. Die Rücktransformation liefert die Aussage von Satz 5.

Schließlich wird noch der Fall der räumlichen achsensymmetrischen Potentialgleichung bei nichtbeschränktem Bereich betrachtet.

Sei B ein nichtbeschränkter, einfach zusammenhängender Bereich der oberen Halbebene $y > 0$ in der x, y-Ebene. Der Rand Γ von B bestehe aus den beiden Abschnitten $x \leqq a$ und $x \geqq b$ der x-Achse und einem stückweise glatten Kurvenstück K, das ganz in der Halbebene $y > 0$ liegt, ausgenommen Anfangspunkt P_a und Endpunkt P_b. K habe die Darstellung $(x(s), y(s))$, $0 \leqq s \leqq l$ mit der Bogenlänge s. K sei von s im mathematisch positiven Sinn durchlaufen (B zur Linken). t und ν sind Tangenten- und innerer Normalvektor auf Γ.

Satz 6. *Sei φ eine Lösung der Differentialgleichung $\frac{1}{y}\left[(y\,\varphi_x)_x + (y\,\varphi_y)_y\right] = 0$ in B. Es gebe positive Konstanten c und ε, so daß $|\operatorname{grad} \varphi| \leqq \frac{c}{r^{2+\varepsilon}}$ in B ist mit $r = \sqrt{x^2 + y^2}$. Auf K sei (mit einer Zahl s_1 aus $0 \leqq s_1 \leqq l$)*

$$-\varphi_\nu \leqq 0 \quad \textit{für} \quad 0 \leqq s \leqq s_1,$$
$$-\varphi_\nu \geqq 0 \quad \textit{für} \quad s_1 \leqq s \leqq l.$$

P_1 sei der dem Parameterwert s_1 entsprechende Punkt von K. Wenn K glatt ist bei P_1, dann gilt

$$\varphi_t \geqq 0 \quad \textit{in } P_1.$$

Für $\varphi = \text{const}$ in $B + \Gamma$ ist der Satz richtig. Für nicht konstantes φ läuft der Beweis analog zum Beweis von Satz 4. Wie dort ist

$$\psi(P) = \psi(P^*) + \int_{P^*}^{P} (g\,dx + f\,dy)$$

mit $f = y\varphi_x$, $g = -y\varphi_y$ nach Vorgabe von $\psi(P^*)$ eine eindeutige, stetige Funktion mit stetigen partiellen Ableitungen erster Ordnung in $B + \Gamma$, solange der Integrationsweg in $B + \Gamma$ verläuft. In B hat ψ noch stetige zweite Ableitungen und erfüllt die elliptische Differentialgleichung

$$y\left(\frac{\psi_x}{y}\right)_x + y\left(\frac{\psi_y}{y}\right)_y = \Delta\psi - \frac{\psi_y}{y} = 0.$$

ψ nimmt also seine untere Grenze bezüglich B in keinem inneren Punkt von B an.

Auf K ist $\psi(P) - \psi(P_1) \geqq 0$ wegen

$$\psi_t = -\varphi_\nu \begin{cases} \geqq 0 & \text{für } s_1 \leqq s \leqq l \\ \leqq 0 & \text{für } 0 \leqq s \leqq s_1. \end{cases}$$

Auf den Abschnitten $x \leqq a$ und $x \geqq b$ der x-Achse gilt $\psi(P) = \psi(P_a)$ bzw. $\psi(P) = \psi(P_b)$ wegen $y = 0$. Also ist

$$\psi(P) \geqq \psi(P_1) \quad \text{für } P \in \Gamma.$$

Für $r(P) \to \infty$ wird $\psi(P) \to \psi(P_b) = \psi(P_a)$ wegen $|\operatorname{grad}\varphi| \leqq \frac{c}{r^{2+\varepsilon}}$, wie man leicht einsieht, wenn man in $\psi(P) - \psi(P_b) = \int_{P_b}^{P} (f\,dx + g\,dy) = \int_{P_b}^{P} y(\varphi_x\,dx - \varphi_y\,dy)$ zunächst ein Stück längst der x-Achse und dann über ein Kreisbogenstück integriert.

Es ist also $\psi_\nu \geqq 0$ in P_1 und wegen $\psi_\nu = y\varphi_t$ und $y > 0$ folgt

$$\varphi_t \geqq 0 \quad \text{in } P_1.$$

Bisher wurde immer gefordert, daß der Rand Γ durch das Vorzeichen des Randausdruckes $L^*[\varphi]$ in zwei Kurvenstücke zerlegt wird. Eine Verallgemeinerung des Satzes 4 ist auch für den Fall möglich, daß Γ durch das Vorzeichen von $L^*[\varphi]$ in eine gerade Anzahl Kurvenstücke mit jeweils wechselndem Vorzeichen zerlegt wird. Auch auf mehrfach zusammenhängende Gebiete kann der Satz erweitert werden, wenn man noch verlangt, daß $\oint L^*[\varphi]\,ds = 0$ ist über jeden Randteil, der eine geschlossene Kurve bildet. Der praktischen Anwendbarkeit dieser Verallgemeinerungen steht allerdings die Tatsache entgegen, daß außer über $L^*[\varphi]$ auch noch Kenntnisse über die Tangentialableitung φ_t an gewissen Randstellen erforderlich sind.

3. Beispiele

3.1. Randwertaufgabe mit nichtlinearer Randbedingung

In Beispiel 4.3 von [*9*] wurde für die Lösung der Randwertaufgabe

$$-\Delta u = 1 \quad \text{in } G\colon\ |x| < 1, |y| < 1; \quad |\operatorname{grad} u| = u \quad \text{auf } \Gamma$$

eine Näherung

$$v = -\frac{x^2 + y^2}{4} + a_0 + a_1 v_1 + a_2 v_2$$

mit $v_k = Re\left[(x + iy)^{4k}\right]$, $a_0 = 0{,}848044$, $a_1 = -0{,}019562$, $a_2 = 0{,}000529$ aufgestellt und die Fehlerabschätzung

$$|u - v| \leqq 0{,}005 \quad \text{in } G + \Gamma$$

durchgeführt. Mit den Formeln von Satz 1 läßt sich damit eine Fehlerabschätzung für die Ableitungen durchführen, die allerdings nur im Inneren von G endliche Schranken liefert.

Sei $R(x, y)$ der Abstand des Punktes $P=(x, y)$ von Γ, d.h.

$$R(x, y) = \operatorname{Min}\left(1 - |x|, 1 - |y|\right).$$

Dann gilt nach (1.2) für die Ableitung in einer beliebigen Richtung:

$$|u_\xi - v_\xi| \leqq \frac{4}{\pi} \cdot 0{,}005 \cdot \frac{1}{R(x, y)} \leqq \frac{0{,}00637}{R(x, y)},$$

also z.B.

$$0{,}247 \leqq -u_x(\tfrac{1}{2}, 0) \leqq 0{,}273.$$

3.2. Temperaturverteilung in einer Platte

Eine Hälfte einer quadratischen Platte werde erwärmt, die andere Hälfte gekühlt. Am Rande wird eine feste Temperatur eingehalten. Unter vereinfachenden Annahmen erhält man für die Temperatur u die Randwertaufgabe:

$$-\Delta u = 6x(1 - x^2) \qquad \text{in } G\colon \qquad |x| < 1, \quad |y| < 1,$$

$$u = 0 \qquad \text{auf } \Gamma\colon \quad \begin{cases} |x| = 1, & |y| \leqq 1, \\ |y| = 1, & |x| \leqq 1. \end{cases}$$

Der Ansatz

$$v = \underbrace{0{,}7x - x^3 + 0{,}3x^5}_{v_0} + \sum_{j=1}^{p} a_j v_j \quad \text{mit} \quad v_j = \frac{\sin j\pi x \cdot \cos j\pi y}{\cosh j\pi}$$

erfüllt bereits die Differentialgleichung in G und die Randbedingungen auf den Randteilen $x = \pm 1$, $|y| \leqq 1$. Eine Fehlerabschätzung für die Lösung selbst liefert der Randmaximumssatz. Es ist

$$|u - v| \leqq \operatorname*{Max}_{\Gamma} |u - v| \quad \text{in ganz } G + \Gamma.$$

Für die Ableitung in beliebiger Richtung folgt damit nach (1.2):

$$|u_\xi - v_\xi| \leqq \frac{4}{\pi \cdot R(x, y)} \operatorname*{Max}_{\Gamma} |u - v|,$$

wo $R(x, y) = \operatorname{Min}\left(1 - |x|, 1 - |y|\right)$ der Abstand vom Rand ist.

Je nachdem, wieviele Ansatzfunktionen man berücksichtigt, erhält man:

	$a_1 = -0{,}2365$	$a_1 = -0{,}2353$ $a_2 = +0{,}0076$	$a_1 = -0{,}2353$ $a_2 = +0{,}00735$ $a_3 = -0{,}00108$
$\lvert u - v\rvert \leqq$	0,0080	0,0011	0,0003
$R(x, y)\,\lvert u_\xi - v_\xi\rvert \leqq$	0,0102	0,0014	0,0004

also $0{,}6360 \leqq u_x(0, 0) \leqq 0{,}6368$.

Gegen den Rand wachsen die Schranken über alle Grenzen. Bei den Randteilen $x = \pm 1$, $|y| \leqq 1$ läßt sich Abhilfe schaffen, wenn man berücksichtigt, daß der Fehler $|u - v|$ auf diesen Randteilen verschwindet. Man kann etwa folgendermaßen verfahren:

Als Gebiet G^* mit Rand Γ^* wird wegen des ungeraden Verhaltens aller Funktionen bezüglich des Nullpunktes der x-Achse das Teilgebiet $0 < x < 1$, $|y| < 1$ von G eingeführt.

Nun werden zwei Näherungen $v^{(1)}$ und $v^{(2)}$ bestimmt mit $u - v^{(1)} \geqq 0$ auf Γ^*, $v_x^{(1)}(1, 0)$ möglichst klein und $u - v^{(2)} \leqq 0$ auf Γ^*, $v_x^{(2)}(1, 0)$ möglichst groß. Dann ist $v^{(1)} \leqq u \leqq v^{(2)}$ in $G^* + \Gamma^*$ oder mit $v = \frac{v^{(1)} + v^{(2)}}{2}$:

$$|u - v| \leqq \frac{v^{(2)} - v^{(1)}}{2}.$$

Sei $P = (x, y)$ mit festem y, $|y| < 1$ und $x = 1 - \varepsilon$, $0 < \varepsilon < 1 - |y|$. Nach (1.2) ist

$$|u_\xi(P) - v_\xi(P)| \leqq \frac{4}{\pi\,\varepsilon} \operatorname*{Max}_{|Q-P| \leqq \varepsilon} |u(Q) - v(Q)| \leqq \frac{4}{\pi\,\varepsilon} \operatorname*{Max}_{|Q-P| \leqq \varepsilon} \left| \frac{v^{(1)}(Q) - v^{(2)}(Q)}{2} \right|.$$

Nun ist, wenn man $P_1 = (1, y)$ einführt,

$$\operatorname*{Max}_{|Q-P| \leqq \varepsilon} |f(Q)| \leqq \operatorname*{Max}_{\substack{|Q-P_1| \leqq 2\varepsilon \\ Q \in G^*}} |f(Q)|$$

und, nach Mittelwertsatz und Schwartzscher Ungleichung

$$|v^{(2)}(Q) - v^{(1)}(Q)| \leqq |Q - P_1| \cdot |\operatorname{grad}(v^{(2)} - v^{(1)})|_{P_1 + \vartheta(Q - P_1)}.$$

Damit erhält man

$$|u_\xi(P) - v_\xi(P)| \leqq \frac{4}{\pi\,\varepsilon} \cdot 2\varepsilon \cdot \frac{1}{2} \operatorname*{Max}_{\substack{|Q-P_1| \leqq 2\varepsilon \\ Q \in G^*}} \left|\operatorname{grad}\left(v^{(2)}(Q) - v^{(1)}(Q)\right)\right|,$$

also für $\varepsilon \to 0$ die Abschätzung auf dem Rand:

$$|u_\xi(1, y) - v_\xi(1, y)| \leqq \frac{4}{\pi} \left|\operatorname{grad}\left(v^{(2)}(1, y) - v^{(1)}(1, y)\right)\right|.$$

Für den Ansatz mit zwei Parametern $v^{(i)} = v_0 + a_1^{(i)} v_1 + a_2^{(i)} v_2$, $i = 1, 2$ ist

$$|u - v_0 + 0{,}23655\, v_1 - 0{,}007525\, v_2| \leqq 0{,}00225\, v_1 - 0{,}000425\, v_2,$$

also

$$|u_\xi - v_{0\xi} + 0{,}23655\, v_{1\xi} - 0{,}007525\, v_{2\xi}| \leqq \frac{8}{\pi} \cdot |0{,}00225\, v_{1x} - 0{,}000425\, v_{2x}|$$

in den Randpunkten $P_1 = (1, y)$. Für $y = 0$ z. B. erhält man

$$0{,}7342 \leqq -u_x(1, 0) \leqq 0{,}7373.$$

3.3. Umströmung eines Rotationsellipsoids

Das Geschwindigkeitspotential Φ einer achsensymmetrischen räumlichen Strömung um ein Rotationsellipsoid genügt unter vereinfachenden Annahmen der Randwertaufgabe

$$\begin{array}{lll} -\Delta\Phi = 0 & \text{in } B\colon & \frac{x^2}{2} + y^2 + z^2 > 1, \\ -\frac{\partial\Phi}{\partial\nu} = 0 & \text{auf } \Gamma\colon & \frac{x^2}{2} + y^2 + z^2 = 1, \end{array} \qquad \lim_{r \to \infty} \left(\frac{\partial\Phi}{\partial x} - 1\right) = 0.$$

Dabei ist ν die innere Normale auf Γ und $\varrho^2 = y^2 + z^2$, $r^2 = x^2 + \varrho^2$. Durch Wahl geeigneter Koeffizienten a_j läßt sich eine Näherung

$$V = x + \sum_{j=1}^{p} a_j V_j$$

mit $V_1 = \frac{x}{r^3}$, $V_2 = \frac{x}{r^7}(4x^2 - 6\varrho^2), \ldots$ aufstellen.

Mit Hilfe des Randmaximumssatzes und der Transformation durch reziproke Radien hat L. COLLATZ [2] eine Abschätzung angegeben von der Form

$$|V - \Phi| \leqq \frac{a}{r}$$

mit einer Konstanten a, die vom Defekt in der Randbedingung abhängt. Damit läßt sich auch gleich wieder eine Fehlerabschätzung für die Ableitungen nach Nr. 1 angeben. Man erhält

$$|V_\xi - \Phi_\xi| \leqq \frac{3a}{2d(r-d)},$$

wobei $d = \mathrm{Min}[R(P), r/2]$ ist mit dem Abstand $R(P)$ von Γ.

Diese Schranken lassen sich nicht auf den Rand ausdehnen. Andererseits interessiert hier besonders die Tangentialgeschwindigkeit am Profilrand.

Wenn man Zylinderkoordinaten x, ϱ, ϑ einführt und die Achsensymmetrie berücksichtigt, erhält man für Φ eine Randwertaufgabe mit den in Satz 6 von Nr. 2 geforderten Eigenschaften.

$$\frac{1}{\varrho}\left[(\varrho\Phi_x)_x + (\varrho\Phi_\varrho)_\varrho\right] = 0 \quad \text{in } B^*\colon \quad \varrho > 0, \quad \frac{x^2}{2} + \varrho^2 > 1,$$

$$\Phi_\nu = 0 \quad \text{auf } K\colon \quad \varrho \geqq 0, \quad \frac{x^2}{2} + \varrho^2 = 1, \quad \lim_{r\to\infty}\left(\frac{\partial\Phi}{\partial x} - 1\right) = 0.$$

Sei $(\Phi_x - 1) = O(1/r^3)$ und $\Phi_y = O(1/r^3)$ für $r \to \infty$. Wegen $\operatorname{grad} V_i = O(1/r^3)$ ist auch $\operatorname{grad}\varphi = O(1/r^3)$ für $\varphi = V - \Phi = x + \sum_{j=1}^{p} a_j V_j - \Phi$. Zur Durchführung einer Abschätzung für die Tangentialableitung im Punkt P_1 mit $x = 0$, $\varrho = 1$ werden zwei Näherungen

$$V^{(i)} = x + a_1^{(i)} V_1 + a_2^{(i)} V_2 \qquad (i = 1, 2)$$

aufgestellt, so daß für $\varphi^{(i)} = V^{(i)} - \Phi$ auf Γ gilt

$$-\varphi_\nu^{(1)} \geqq 0, \qquad -\varphi_\nu^{(2)} \leqq 0 \quad \text{für} \quad x \leqq 0,$$

$$-\varphi_\nu^{(1)} \leqq 0, \qquad -\varphi_\nu^{(2)} \geqq 0 \quad \text{für} \quad x \geqq 0.$$

Dann ist

$$\varphi_t^{(1)} \leqq 0 \leqq \varphi_t^{(2)} \quad \text{in } P_1,$$

d.h.

$$V_t^{(1)} \leqq \Phi_t \leqq V_t^{(1)} \quad \text{in } P_1.$$

Man erhält z.B. $V^{(1)} = x + 0{,}35\,V_1 + 0{,}017\,V_2$, $V^{(2)} = x + 1{,}08\,V_1 + 0{,}088\,V_2$ und damit die Schranken

$$1{,}248 \leqq \Phi_t \leqq 1{,}552 \quad \text{im Punkt } P_1 \quad (x = 0,\ \varrho = 1).$$

Diese Arbeit ist ein Teilauszug aus der Dissertation des Verfassers. Ein anderer Teilauszug wurde in [9] gegeben.

Literatur

[*1*] Collatz, L.: Numerische Behandlung von Differentialgleichungen, 2. Aufl. Berlin-Göttingen-Heidelberg 1955. (Engl. Ausgabe: 3. Aufl. 1960.)

[*2*] Collatz, L.: Fehlerabschätzungen bei Randwertaufgaben partieller Differentialgleichungen mit unendlichem Grundgebiet. Z. angew. Math. Phys. **9a**, 118–128 (1958).

[*3*] Courant, R., & D. Hilbert: Methoden der Mathematischen Physik. Bd. 1, 2. Aufl., Berlin 1931. Bd. 2, Berlin 1937.

[*4*] Douglis, A., & L. Nirenberg: Interior estimates for elliptic systems of partial differential equations. Comm. Pure Appl. Math. **8**, 503–538 (1955).

[*5*] Duff, G. F. D.: Partial Differential Equations. Toronto 1956.

[*6*] Hartman, Ph., & A. Wintner: On the local behavior of solutions of non-parabolic partial differential equations. Amer. J. Math. **75**, 449–476 (1953).

[*7*] Hopf, E.: Über den funktionalen, insbesondere den analytischen Charakter der Lösungen elliptischer Differentialgleichungen zweiter Ordnung. Math. Z. **34**, 194–233 (1932).

[*8*] Kellogg, O. D.: Foundations of Potential Theory. Berlin 1929.

[*9*] Meyer, A. G.: Schranken für die Lösungen von Randwertaufgaben mit elliptischer Differentialgleichung. Arch. Rational Mech. Anal. **6**, 277–298 (1960).

[*10*] Schauder, J.: Über lineare elliptische Differentialgleichungen zweiter Ordnung. Math. Z. **38**, 257–282 (1934).

[*11*] Schauder, J.: Numerische Abschätzungen in elliptischen linearen Differentialgleichungen. Studia math. **5**, 34–42 (1934).

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Bemerkung

zur Arbeit H. Ehrmann, Iterationsverfahren mit veränderlichen Operatoren, Vol. 4, 45–64 (1959).

Herr J. Schröder aus Hamburg machte mich freundlicherweise darauf aufmerksam, daß das Zusatzaxiom (2.1 g) auf S. 47 eine häufig nicht erfüllte Forderung darstellt.

Unter der Voraussetzung B 1 a genügt es jedoch, stets gleichmäßige Konvergenz zu fordern, also (2.1 g) wie folgt abzuschwächen:

Aus $\lim \varrho_n = 0$ und $0 \leqq \varrho_n$ für $n = 1, 2, \ldots$ folgt die Existenz eines (festen) Elementes τ derart, daß für jede positive Zahl k ein $n_1(k)$ existiert mit $\varrho_n \leqq \tau/k$ für alle $n \geqq n_1(k)$.

Der Beweis bleibt dabei unverändert.

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